Approximating Artinian Rings by Gorenstein Rings
and
3-Standardness of the Maximal Ideal

By

Ananthnarayan Hariharan

Submitted to the Department of Mathematics and the
Faculty of the Graduate School of the University of Kansas
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Committee members

Craig Huneke, Chairperson

Margaret Bayer

John Bricke

Daniel Katz

Satyagopal Mandal

Date defended: 16th July 2009

The Dissertation Committee for Ananthnarayan Hariharan certifies
that this is the approved version of the following dissertation:

Approximating Artinian Rings by Gorenstein Rings
and
3-Standardness of the Maximal Ideal

Committee:

Craig Huneke, Chairperson

Margaret Bayer

John Bricke

Daniel Katz

Satyagopal Mandal

Date approved: 16th July 2009

Approximating Artinian Rings by Gorenstein Rings

and 3-Standardness of the Maximal Ideal

Ananthnarayan Hariharan

University of Kansas

Advisor: Craig Huneke

July 2009

Abstract

We study two different problems in this dissertation. In the first part, we wish to understand how one can approximate an Artinian local ring by a Gorenstein Artin local ring. We make this notion precise in Chapter 2, by introducing a number associated to an Artin local ring, called its Gorenstein colength . We study the basic properties and give bounds on this number in this chapter. We extend results due to W. Teter, C.Huneke and A. Vraciu by studying the relation of Gorenstein colength with self-dual ideals. In particular, we also answer the question as to when the Gorenstein colength is at most two.

In Chapter 3, we show that there is a natural upper bound for Gorenstein colength of some special rings. We compute the Gorenstein colengths of these rings by constructing some Gorenstein Artin rings. We further show that the Gorenstein colength of Artinian quotients of two-dimensional regular local rings are also bounded above by the same upper bound by using a formula due to Hoskin and Deligne.

Given two Gorenstein Artin local rings, L. Avramov and W. F. Moore construct another Gorenstein Artin local ring called a connected sum. We use this to improve a result of C. Huneke and A. Vraciu in Chapter 4. We also define the notion of a connected sum more generally and apply it to give bounds on the Gorenstein colengths of fibre products of Artinian local rings.

In the second part of the thesis, we study a notion called n-standardness of ideals primary to the maximal ideal in a Cohen-Macaulay local ring. We first prove the equivalence of n-standardness to the vanishing of a certain Koszul homology module up to a certain degree. We go over the properties of Koszul complexes and homology needed for this purpose in Chapter 5.

In Chapter 6, we study conditions under which the maximal ideal is 3-standard. We first prove results when the residue field is of prime characteristic and use the method of reduction to prime characteristic to extend the results to the characteristic zero case. As an application, we see that this helps us extend a result due to T. Puthenpurakal in which he shows that a certain length associated to a minimal reduction of the maximal ideal does not depend on the minimal reduction chosen.

Acknowledgements

The journey of getting a PhD is a long arduous one with many pitfalls and many-a-moment of elation thrown in together. What made it possible for me to reach the destination is the people around me, who helped me get through the moments of frustration and shared the moments of joy. I would like to thank everyone who has contributed in ways big or small that helped me stay the course.

Firstly, I would like to thank the Mathematics Department at the University of Kansas for having put up with me for seven years. The faculty members have been very accessible, the office staff very friendly and the graduate students very supportive. In particular, special thanks to Samantha, Gloria, Debbie, Kerrie, Pam and Joan among the office staff for making life easier for us; to professors Gavosto, Porter and Torres, whose presence ensured that we had a smooth sailing; to Alex, Aslihan, Bahman, Branden, Christy, Dan, Erika, Erin, Forrest, Giulio, Janet, Jarod, Jila, Jyoti, Kabe, Lon, Lucas, Lynne, Manoj, Melissa, Nate, Nick, Nicole, Perneet, Tom and William for being such great friends and colleagues (and to the Denominators: Alex, Erika, Kabe and Lynne for maintaining the daily Wescoe run throughout our graduate student careers :). Thanks to Prof. Gavosto for being a sounding board and all her advise when it came to teaching and to Purna for having endured us in reading courses for six semesters.

Secondly, I would like to thank the algebra community for guiding me and helping me improve as a mathematician through these years. In particular, special thanks to professors Avramov, Migliore, Rossi, Srinivasan and Ulrich and to Adela, Anurag, Claudia, Diana, Dumi, Erin, Graham, Frank, Florian, Kazuma, Lars, Laura, Liana, Livia, Louiza, Nick, Olga, Sean, Srikanth, Trung, Wenliang and Yongwei for all the pleasant interactions, both mathematical and personal, during various conferences and schools. The algebra and combinatorics groups at KU have been highly instrumental in helping this greenhorn find my niche, and in these later years, hone my skills. I really appreciate the efforts of Craig, Dan, Javid, Jeff, Jeremy, Krishna, Long, Marge, Purna and Satya in helping me reach this stage.

Thirdly, I would like to thank the Indian community here at KU. Their support was necessary to get adjusted and stay steady throughout these years. Every game of cards, movie, picnic, get-together and potluck helped in this respect. Special thanks to Nikunj, GeetAnshu, Diptesh, Ritesh, Rohita, Anand, Sudarshan, Santosh, Varsha, Amruta, Shyam, Bhargav, Harish, Vamsi, Lakshmi, Lakshmy, Vini, Suddu and Raghavi, Ramu and Suji, Ramya and Abhijith among the folks currently in Lawrence and Angal, Dumre, Badri, Aravind, Visu, Hari, Rajesh, Sudee, Guna and Madhuri, Santosh, Raghavendra, Charuta, Abhishek, Supreetha, Vara, Abdul, Dinesh and Rajaneesh among the folks whom I met in Lawrence and are now spread out in other parts of the world.

I would like to appreciate the efforts of Craig, Dan, Srikanth, Adela, Estela and Lindsey in helping me with my job search and would also like to thank my Ph.D. thesis committee, Craig, Dan, Satya, Marge and John Bricke.

Some special friends whom I have known for years and without whom it would have been an impossible task are Sachin and Roopa, Giri, Vini, Danny, Vidya, Jayan, Manoj, Dishant and Suna, Sumedha, Meera, Vivek, Sooraj and Venky. In particular, there are three people whom I must mention: Rekha, without whom I would never have travelled so far away from home; Christy, who made it much more easier to move into a new country and understand a new culture; and Badri (and now Smitha), many discussions with whom have ensured that I have not lost my identity in a new world.

I would like to acknowledge my *Gurus*, who equipped me with the tools necessary to embark on my quest for a Ph.D. Thanks to Divate Sir, for pushing us when we were beginners during the B.Sc. days, thanks to Prof. Kumaresan for running the MTTS programme and teaching us how to think, thanks to Prof. Jugal Verma, Prof. Ravi Rao and Prof. Balwant Singh for teaching me commutative algebra and helping me get to Kansas.

One does not have to look around much to find a great researcher, or an inspiring teacher, or an amazing human being. But only a few people are all of the three. I have been fortunate enough to work with, and have as an advisor, one such person in Craig Huneke. What I have learnt from him is a lot more than just mathematics. I have grown a lot as a person by just being around, observing and learning from him. I would like to express my heartfelt gratitude for his guidance and for everything he has taught me.

None of this would have been possible, nor would any of this be meaningful without the blessings and support of my family. Words can neither express how much my *PaaTi*, *Amma, Appa* and sister mean to me, nor are they sufficient to express my gratitude for everything they have done. I thank them for just being who they are. I dedicate this thesis to my grandfather, the mathematics school teacher, who has always been an inspiration and a guiding light.

Introduction

This dissertation focuses mainly on two probelms. In the first half, we use the notion of Gorenstein colength (defined in this thesis) to approximate an Artinian local ring by a Gorenstein Artin local ring. In the second part, we study consequences of n-standardness of ideals (defined by M. Rossi in [20]) which are primary to the maximal ideal in a local ring. We further explore 3-standardness of the maximal ideal using characteristic p methods.

A] Approximating Artinian local rings by Gorenstein Artin local rings:

A problem of interest in commutative algebra is to find Gorenstein local rings S mapping onto a given Cohen-Macaulay local ring R. In particular, let T be a commutative Noetherian local ring and $\mathfrak{b}$ an ideal in T such that $R := T/\mathfrak{b}$ is Cohen-Macaulay. One would like to find ideals $\mathfrak{c} \subseteq \mathfrak{b}$ such that $S := T/\mathfrak{c}$ is Gorenstein. In this case, S is a Gorenstein local ring mapping onto R.

We are interested not only in finding such a Gorenstein ring, but also in finding one as "close" to R as possible. More specifically, the question we would like to answer is the following: Given an Artinian local ring $(R, \mathfrak{m}, \mathsf{k})$, how "close" can one get to R by a Gorenstein Artin local ring?

In this dissertation, in order to make the notion of approximating an Artinian local ring by a Gorenstein Artin local ring precise, we introduce a number called the Gorenstein colength of R, denoted $g(R)$. The number $g(R)$ gives a numerical value to how close one can get to an Artinian local ring R by a Gorenstein Artin local ring. It can be easily seen that $g(R)$ is finite.

The problem with studying Gorenstein colength is two-fold, which can be summarized in the following questions: Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring with canonical module ω. The main questions one would like to answer are the following:

a) How does one intrinsically compute $g(R)$?

b) How does one construct a Gorenstein Artin local ring S mapping onto R such that $\lambda(S) - \lambda(R) = g(R)$?

To answer the first question, we exhibit some natural bounds on the Gorenstein colength of R. The main set of inequalities are the following:

$$\boxed{\lambda(R/(\omega^*(\omega))) \leq \quad \min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\} \quad \leq \quad g(R) \quad \leq \quad \lambda(R).}$$

where by $(\_)^*$ and $(\_)^\vee$, we mean $\mathrm{Hom}_R(\_, R)$ and $\mathrm{Hom}_R(\_, \omega)$ respectively, $\omega^*(\omega)$ is the trace ideal of ω and $\lambda(\_)$ denotes length.

As for the second question, we resort to different techniques to construct Gorenstein Artin rings in different contexts. A useful construction is the idea of a connected sum of two Gorenstein Artin local rings S_1 and S_2 over another Gorenstein Artin local ring R. This newly constructed ring is also a Gorenstein Artin local ring whose length can be determined in terms of the lengths of S_1, S_2 and R. With some mild hypothesis, this helps us get an upper bound on the Gorenstein colengths of the fibre product of two Artinian local rings R_1 and R_2 over R.

A well-known method of constructing Gorenstein Artin local rings is the following: If $(S, \mathfrak{m}_S, \mathsf{k})$ is a Gorenstein Artin local ring, then so is $S/(0 :_S f)$ for any $f \in S$. We use this technique and a result of Reid, Roberts and Roitman ([19]) to determine the Gorenstein colength of $T/\mathfrak{m}_T^n$, where $T = \mathsf{k}[|x_1, \ldots, x_d|]$ is a power series ring over a field k of characteristic zero and $\mathfrak{m}_T = (x_1, \ldots, x_d)$.

With the result for $T/\mathfrak{m}_T^n$, using the technique of a flat base change, we prove the following in [2]:

Theorem 0.1. *Let $T = \mathsf{k}[|x_1, \ldots, x_d|]$ be a power series ring over a field k of characteristic zero and $\mathfrak{c} = (f_1, \ldots, f_d)$ be an ideal generated by a system of parameters. Then $g(T/\mathfrak{c}^n) = \lambda(T/\mathfrak{c}^{n-1})$.*

One can use the notion of a connected sum over k and give a different characteristic-free proof for the $n = 2$ case (see Theorem 4.27).

When R is a codimension 2 Artinian local ring, i.e., $R \simeq T/\mathfrak{b}$, where T is a two-dimensional regular local ring, we use a formula of Hoskin and Deligne to show that $g(R) \leq \lambda(R/\mathrm{soc}(R))$.

It can be seen from the definition that $g(R)$ is zero if and only if R is Gorenstein and $g(R) = 1$ if and only if R is not Gorenstein and $R \simeq S/\mathrm{soc}(S)$ for a Gorenstein Artin ring S. W. Teter gives a characterization for such rings in his paper [22]. In their paper [12], C. Huneke and A. Vraciu refer to these rings as *Teter's rings*. Teter's theorem states:

Theorem 0.2 (Teter)**.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian ring with canonical module ω. Then the following are equivalent:*

i) $g(R) \leq 1$.

ii) *Either R is Gorenstein or there is an isomorphism $\mathfrak{m} \xrightarrow{\phi} \mathfrak{m}^\vee$ such that $\phi(x)(y) = \phi(y)(x)$, for every x, y in $\mathfrak{m}$, where by $(\_)^\vee$, we mean $\mathrm{Hom}_R(\_, \omega)$.*

The commutativity condition on the map ϕ in (ii) of Theorem 0.2 is an awkward technical condition. The following theorem ([12], Theorem 2.5), of Huneke and Vraciu is an improvement of Theorem 0.2, which gets rid of Teter's technical condition on the map ϕ. However, they need to assume that 2 is invertible in R and $\operatorname{soc}(R) \subseteq \mathfrak{m}^2$.

Theorem 0.3 (Huneke-Vraciu)**.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian ring such that $1/2 \in R$,* $\operatorname{soc}(R) \subseteq \mathfrak{m}^2$. *With notations as in Teter's theorem, the following are equivalent:*

i) $g(R) \leq 1$.

ii) *Either R is Gorenstein or* $\mathfrak{m} \simeq \mathfrak{m}^\vee$.

Using connected sums over k, we show that the condition on the socle of R is not necessary for the above result to hold true. This gives a characterization of Artinian local rings R containing $1/2$ with $g(R) = 1$ (see Theorem 4.29).

When R contains a coefficient field, Huneke and Vraciu construct a Gorenstein Artin local ring S mapping onto R such that $\lambda(S) - \lambda(R) = 1$, thus giving a different proof for the above theorem. We show that if $R/\mathfrak{a}$ is an algebra retract of R and $\mathfrak{a} \simeq \mathfrak{a}^\vee$, then with some further assumptions we can conclude that $g(R) \leq \lambda(R/\mathfrak{a})$, thus extending their theorem.

A natural question one can ask is whether we can characterize Artinian local rings whose Gorenstein colength is two. We extend Teter's theorem and the Huneke-Vraciu theorem and as a corollary, prove the following in [1]:

Theorem 0.4. *Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring and $\mathfrak{b}$ an $\mathfrak{m}_T$-primary ideal such that $\mathfrak{b} \subseteq \mathfrak{m}_T^6$. Moreover, assume that 2 is invertible in $R := T/\mathfrak{b}$. Then the following are equivalent:*

i) $g(R) \leq 2$.

ii) *There exists an ideal $\overline{\mathfrak{a}} \subseteq R$, $\lambda(R/\overline{\mathfrak{a}}) \leq 2$ such that $\overline{\mathfrak{a}} \simeq \overline{\mathfrak{a}}^\vee$, where $\bar{}$ denotes going modulo $\mathfrak{b}$.*

The main questions we try to answer in this thesis are the following: Let $(R,\mathfrak{m},\mathsf{k})$ be an Artinian local ring with canonical module ω.

Question 0.5. Is $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^{\vee}\} = g(R)$?

A stronger question one can ask is:

Question 0.6. Given an $\mathfrak{a}$ in R such that $\mathfrak{a} \simeq \mathfrak{a}^{\vee}$, is there an Artinian Gorenstein local ring S such that $\lambda(S) - \lambda(R) = \lambda(R/\mathfrak{a})$?

Let the hypothesis as in Theorem 0.4. By combining the conclusions of the Huneke-Vraciu theorem and Theorem 0.4, we see that $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^{\vee}\} = g(R)$ when either of the two quantities is at most two. Further, it follows from Theorem 0.4 and the main inequalities that if $g(R) = 3$, then so is $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^{\vee}\}$. Thus we see that in this case, Question 0.5 has a positive answer if either $g(R) \leq 3$ or $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^{\vee}\} \leq 2$.

B] 3-Standardness of the maximal ideal:

Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with infinite residue field k. Let I be an $\mathfrak{m}$-primary ideal and $J = (x_1,\ldots,x_d)$ be a minimal reduction of I. In [23], P. Valabrega and G. Valla show that the condition $I^n \cap J = JI^{n-1}$ holds for all n if and only if the associated graded ring $\mathrm{gr}_R(I) = R/I \oplus I/I^2 \oplus \cdots$ is Cohen-Macaulay.

In [20], M. Rossi studies the condition $J \cap I^k = JI^{k-1}$ for all $k \leq n$. She calls such ideals n-standard. In this part of the dissertation we study some consequences of n-standardness and explore when the maximal ideal is 3-standard.

In [18], T. Puthenpurakal shows that $\lambda(\mathfrak{m}^3/J\,\mathfrak{m}^2)$ is independent of the minimal reduction J of $\mathfrak{m}$ when R is Cohen-Macaulay by proving the following:

Theorem 0.7 (Puthenpurakal)**.** *Let $(R,\mathfrak{m},\mathsf{k})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with infinite residue field k. If J is a minimal reduction of $\mathfrak{m}$, then*

$$\lambda(\mathfrak{m}^3/J\mathfrak{m}^2) = e_0(R) + (d-1)\mu(\mathfrak{m}) - \mu(\mathfrak{m}^2) - \tbinom{d-1}{2}.$$

In Chapter 6, we extend this result to the nth power of an n-standard $\mathfrak{m}$-primary ideal. We go over the properties of Koszul complexes and homology needed for this purpose in Chapter 5. We first prove the equivalence of n-standardness to the vanishing of a certain Koszul homology module up to a certain degree in Proposition 5.10. Using this, we can prove the following theorem:

Theorem 0.8. *Let $(R,\mathfrak{m},\mathsf{k})$ be a Cohen-Macaulay local ring with an infinite residue field k. If I is an $\mathfrak{m}$-primary ideal and J is a minimal reduction of I such that $J \cap I^i = JI^{i-1}$ for $1 \leq i \leq n$, then*

$$\lambda(I^{k+1}/JI^k) = e_0(I) + \sum_{i=0}^{k} (-1)^{i+1} \binom{d-1}{i} \lambda(I^{k-i}/I^{k-i+1}). \quad \textit{for } 0 \leq k \leq n$$

When $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field k and J is a minimal reduction of $\mathfrak{m}$, it is well-known that $\mathfrak{m}^2 \cap J = J\mathfrak{m}$, for example see Proposition 8.3.3(1) in [21]. Thus $\mathfrak{m}$ is 3-standard if and only if $\mathfrak{m}^3 \cap J = J\mathfrak{m}^2$. In Chapter 6, we investigate conditions under which the equality $J \cap \mathfrak{m}^3 = J\mathfrak{m}^2$ holds for every minimal reduction J of $\mathfrak{m}$.

In Theorems 6.7 and 6.10, we show that $\mathfrak{m}$ is 3-standard when $\mathrm{char}(\mathsf{k}) = p > 0$ in two cases: (i) when k is algebraically closed and the graded ring G associated to the maximal ideal $\mathfrak{m}$ is reduced and connected in codimension 1 and (ii) when k is perfect and G is a normal domain. In order to prove these theorems, we borrow some tools like tight closure (e.g., see [7]) and graded absolute integral closures (e.g., see [8], section 5) from the characteristic $p > 0$ world, in particular, we use the following two theorems:

Theorem 0.9 (Huneke-Vraciu)**.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be an excellent normal local domain, k a perfect field such that $\operatorname{char}(\mathsf{k}) = p > 0$ and reduced associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$. If I is an ideal such that $I \in \mathfrak{m}^k$, then $I^* \subseteq I + \mathfrak{m}^{k+1}$, where I^* is the tight closure of I.*

Theorem 0.10 (Hochster-Huneke)**.** *Let G be a standard graded domain over a field k of characteristic $p > 0$. Let $\mathfrak{G}$ be the graded absolute integral closure of G. Then every sequence that is a part of a homogeneous system of parameters in G forms a regular sequence in $\mathfrak{G}$.*

Theorem 0.9 ([11], Theorem 3.1) is used in the proof of Theorem 6.7 and Theorem 0.10 ([8], Theorem 5.15) is used to prove Theorem 6.10.

We can then use the method of reduction to prime characteristic (e.g., see [9], sections 2.1 and 2.3) to conclude that if $(R, \mathfrak{m}, \mathsf{k})$ is a d-dimensional Cohen-Macaulay local ring with associated graded ring $G = \operatorname{gr}_R(\mathfrak{m}) = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \cdots$ and $J = (x_1, \ldots, x_d)$ is a minimal reduction of $\mathfrak{m}$, then $J \cap \mathfrak{m}^3 = J\mathfrak{m}^2$ for every minimal reduction J of $\mathfrak{m}$ when G is an absolute domain. (See Theorem 6.16).

Thus we see that if G is an absolute domain, then $J \cap \mathfrak{m}^3 = J\mathfrak{m}^2$ for every minimal reduction J of $\mathfrak{m}$, i.e., $\mathfrak{m}$ is 3-standard. As a consequence, we see that in this case, if R is a Cohen-Macaulay local ring with an infinite residue field, the formula $\lambda(\mathfrak{m}^4/J\mathfrak{m}^3) = e_0(\mathfrak{m}) + \sum_{i=0}^{3}(-1)^{i+1}\binom{d-1}{i}\lambda(\mathfrak{m}^{3-i}/\mathfrak{m}^{4-i})$ holds for every minimal reduction J of $\mathfrak{m}$. Therefore, in this case, $\lambda(\mathfrak{m}^4/J\mathfrak{m}^3)$ is independent of the minimal reduction of $\mathfrak{m}$.

Contents

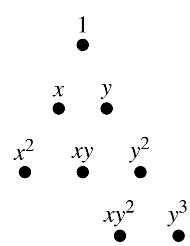
Chapter 1

Foundation

1.1 An Informal Introduction to Gorenstein Artin Rings

The main object of study of this part of the thesis is a class of rings called Gorenstein rings. With the help of a few examples, let us see what Gorenstein rings are.

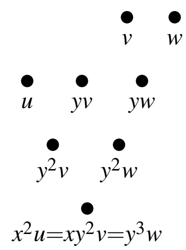
Example 1.1. Let $R = T/I$ where $T = \mathsf{k}[X,Y]$ and $I = (X^3, Y^4, X^2Y, XY^3)$. By x and y, we denote the respective images of X and Y in R. The set $\{1, x, y, x^2, xy, y^2, xy^2, y^3\}$ is a k-basis for R. We can draw the following picture to represent R:



The horizontal rows represent the degrees of the monomials in R. Multiplying a monomial by x takes it into the next row to the left and multiplying by y takes it into the next row to the right.

Associated to an Artinian local ring R, there is a module ω_R called its *canonical module*. The canonical module is “the ring flipped upside down.”

When we flip the picture upside down, we get



which represents

$$\omega_R \simeq \frac{Ru \oplus Rv \oplus Rw}{(xu, yu - x^2v, yv - xw, yw)}.$$

It is clear from the picture that R and ω_R are not isomorphic to each other. Observe that in this case ω_R is generated by the three elements u, v and w as an R-module.

Example 1.2. Let $R = T/I$ where $I = (X^3, Y^4)$. In this case we have R =

$$\begin{array}{c} 1 \\ x \quad y \\ x^2 \quad xy \quad y^2 \\ x^2y \quad xy^2 \quad y^3 \\ x^2y^2 \quad xy^3 \\ x^2y^3 \end{array}$$

When we flip the structure, we see that $\omega_R \simeq Ru/(x^3u, y^4u)$. But $x^3 = y^4 = 0$ in R, hence $\omega_R \simeq R$.

This is an example of a Gorenstein Artin ring.

1.2 Artinian and Gorenstein Artin Rings: Basic Facts

For the proofs of the following facts, one can refer [4] or [5].

Notation:

In tis section, by $(R, \mathfrak{m}, \mathsf{k})$ we mean R is an Artinian local ring with unique maximal ideal $\mathfrak{m}$, residue field k.[1] Let ω_R be a canonical module R. For any R- module M, by $M^\vee$ we mean $\mathrm{Hom}_R(M, \omega_R)$.

1. The Canonical Module

- $\mathrm{ann}_R(\omega_R) = 0$, i.e. ω_R is a faithful R-module.
- $\lambda(R) = \lambda(\omega_R)$.
- $\mathrm{Hom}_R(\mathsf{k}, \omega_R) = \mathrm{soc}(\omega_R) \simeq \mathsf{k}$.
- $\omega_R \simeq E$, the injective hull of k over R.
- $\mathrm{Hom}_R(\omega_R, \omega_R) \simeq R$.
- For any ideal I in R, $\omega_{R/I} = 0 :_{\omega_R} I$.

2. Matlis Duality:

- $^\vee$ preserves short exact sequences (contravariantly).
- If M is an R-module, then $\lambda(M) = \lambda(M^\vee)$. This follows from an induction on $\lambda(M)$, starting with $\mathsf{k}^\vee = \mathrm{Hom}_R(\mathsf{k}, E) \simeq E_{\mathsf{k}}(\mathsf{k}) \simeq E$.
- The natural map $M \to M^{\vee\vee}$ is an isomorphism.

Comment: $^\vee$ basically flips the structure of the module, $M^\vee$ is looking at M upside down.

[1] Some of the statements in this section are also valid in a more general setting, but we are only interested in the Artinian case.

3. Gorenstein Rings

Definition 1.3. *Let $(R,\mathfrak{m},\mathsf{k})$ be an Artinian local ring with canonical module ω. The following are equivalent:*

(1) $\mathrm{id}_R(R) < \infty$.

(2) *R is injective (as a module over itself).*

(3) $R \simeq \omega$.

(4) $\mathrm{soc}_R(R)$ *is a 1-dimensional* k*-vector space.*

(5) *The ideal* (0) *in R is irreducible.*

(6) *For every ideal I in R,* $0 :_R (0 :_R I) = I$.

When any one (and hence all) of the above conditions are satisfied, we say that R is a Gorenstein Artin local ring.

In general, we say that a Noetherian local ring S is Gorenstein if S is Cohen-Macaulay and $S/(x_1,\ldots,x_d)$ is a Gorenstein Artin local ring for some (equivalently for all) system of parameters $x_1,\ldots,x_d$ in S. We have the following:

- A regular local ring $(T,\mathfrak{m},\mathsf{k})$ is Gorenstein. This is true since $\mathfrak{m}$ is generated by a system of parameters , $T/\mathfrak{m} = \mathsf{k}$ is self-injective and T is Cohen-Macaulay.

- If x is a non-zerodivisor on S, then S is Gorenstein if and only if $S/(x)$ is Gorenstein.

- A complete intersection ring (i.e. a quotient of a regular local ring by a regular sequence) is Gorenstein.

- A Noetherian local ring $(S,\mathfrak{m},\mathsf{k})$ is Gorenstein if and only if $\widehat{S}$, the $\mathfrak{m}$-adic completion of S, is Gorenstein.

- Let $(R,\mathfrak{m},\mathsf{k})$ be an Artinian local ring with canonical module ω. Let I be an ideal in R. The following are equivalent:

1. R/I is Gorenstein.

2. $0 :_\omega I \simeq R/I$.

3. $0 :_\omega I$ is a cyclic R/I-module.

4. There is a nonzero element u in ω such that $0 :_R u = I$.

- Let ω be the canonical module of R. We can define a Gorenstein Artin ring structure on $S := R \oplus \omega$ using Nagata's principle of idealization, denoted by $S = R \ltimes \omega$. We define the following operations on S:

$$(r,u) + (s,v) = (r+s, u+v) \text{ and } (r,u) \cdot (s,v) = (rs, rv+su).$$

  Note that ω is an ideal in S and that $\omega^2 = 0$. Hence $\mathfrak{m}_S := \mathfrak{m} \oplus \omega$, the unique maximal ideal in S, is nilpotent, showing that S is Artinian.

  Since $\omega^2 = 0$ in S and ω is a faithful R-module, $\operatorname{ann}_S(\omega) = \omega$ forcing $\operatorname{soc}(S) \subseteq \omega$. Thus $\operatorname{soc}(S) = \operatorname{soc}(\operatorname{soc}(S)) \subseteq \operatorname{soc}(\omega)$. But $\operatorname{soc}(S) \neq 0$ (since S is zero-dimensional) and $\operatorname{soc}(\omega)$ is a 1-dimensional k-vector space, which implies that $\operatorname{soc}(S)$ is also 1-dimensional. This proves that S is Gorenstein.

4. Free Resolutions of Gorenstein Artin Quotients of Regular Local Rings

Set-up: Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring. By Cohen's Structure Theorem, there is a regular local ring $(T, \mathfrak{m}_T, \mathsf{k})$ such that $R = T/I$ for some $\mathfrak{m}_T$-primary ideal I in T. Let $\dim(T) = d$. Since $\operatorname{depth}(R) = 0$, by the Auslander-Buchsbaum formula, $\operatorname{pd}_T(R) = \operatorname{depth}(T) = d$. Consider a minimal free resolution of R over T:

$$\mathbf{F}_\bullet : \quad 0 \to T^{b_d} \xrightarrow{\phi_d} \cdots \to T^{b_1} \xrightarrow{\phi_1} T \to R. \quad (*)$$

- With notation as above, $b_d = \dim_{\mathsf{k}}(\operatorname{soc}(R))$. This can be seen as follows:

Let $(x_1,\dots,x_d)=\mathfrak{m}_T$. Then the Koszul complex $\mathbf{K}_\bullet(\underline{x};T)$ gives a minimal resolution of k over T. Tensoring $\mathbf{K}_\bullet(\underline{x};T)$ by R, we see that $\mathrm{Tor}_T^d(\mathsf{k},R)\simeq \mathrm{soc}(R)$. On the other hand, one can tensor $(*)$ with k to see that $b_d=\dim_{\mathsf{k}}(\mathrm{Tor}_T^d(\mathsf{k},R))$.

In particular, R is Gorenstein if and only if $b_d=1$.

- If $f_1,\dots,f_d$ is a regular sequence in $\mathfrak{m}_T$, then $R:=T/(f_1,\dots,f_d)$ is Gorenstein, since the Koszul complex $\mathbf{K}_\bullet(f_1,\dots,f_d;T)$ gives a minimal resolution of R over T.

- The canonical module of R, $\omega_R\simeq \mathrm{Coker}(\phi_d^*)$ and has a free resolution

$$0\to T^*\xrightarrow{\phi_1^*}(T^*)^{b_1}\to\cdots\to(T^*)^{b_d}\to\omega_R\to 0 \qquad (**)$$

over T. In particular, $\mathrm{Ext}_T^d(R,T)\simeq\omega_R$.

- $\mu(\omega_R)=b_d$.

5. The Graded Case

Notation: Let k be a field and R be a graded ring R with $R_0=\mathsf{k}$. For a finitely generated graded R-module M, by $h_M(i)$ we mean the k-dimension of the ith graded piece of the ring M and if M is Artinian, $\mathrm{Max}(M):=\max\{i:h_M(i)\neq 0\}$.

Definition 1.4.

1. Let R be a graded Artinian local ring, M a finitely generated graded R-module and $h_i=h_M(i)$. The h-vector of M is $(h_0,h_1,\dots,h_{\mathrm{Max}(M)})$.
2. If R is a graded Artinian local ring with h-vector $(h_0=1,h_1,\dots,h_m)$, we say that the Hilbert series of R, denoted $H(R,t)$, is $\sum_{i=0}^m h_it^i$.

- Let ω be the canonical module of a graded Artinian local R. Then $\omega\simeq\mathrm{Hom}_{\mathsf{k}}(R,\mathsf{k})$.

- The h-vector of ω is $(h_{\mathrm{Max}(R)}, \ldots, h_1, h_0 = 1)$.

- If R is a graded Gorenstein Artin local ring, then the h-vector of R is palindromic, i.e., it satisfies the equations $h_i = h_{\mathrm{Max}(R)-i}$ for $i = 0, \ldots, \mathrm{Max}(R)$.

1.3 Gorenstein Ideals of a Fixed Colength

Notation: Let $T = \mathsf{k}[X_1, \ldots, X_d]$ be a polynomial ring over an algebraically closed field k, $\mathfrak{m} = (X_1, \ldots, X_d)$ be the unique homogeneous maximal ideal, and $\mathfrak{b}$, $\mathfrak{c} \subseteq T$ be $\mathfrak{m}$-primary ideals, such that $\mathfrak{c} \subseteq \mathfrak{b}$. Since $\mathfrak{c}$ is $\mathfrak{m}$-primary, some power of each variable X_i is in $\mathfrak{c}$. Fix a_i, $i = 1, \ldots, d$, such that $X_i^{a_i} \in \mathfrak{c}$ and let $\mathfrak{d} = (X_1^{a_1}, \ldots, X_d^{a_d})$.

We want to study the Gorenstein ideals between $\mathfrak{c}$ and $\mathfrak{b}$. Since such a Gorenstein ideal also contains $\mathfrak{d}$, (and $\mathfrak{d}$ itself is Gorenstein), it must be of the form $(\mathfrak{d} :_T f)$ for some polynomial $f \in T$. Without loss of generality, we may assume that f consists only of monomials not in $\mathfrak{d}$.

Thus, in order to characterize Gorenstein ideals between $\mathfrak{c}$ and $\mathfrak{b}$, it is enough to characterize polynomials in T in monomials not in $\mathfrak{d}$, such that $\mathfrak{c} \subseteq (\mathfrak{d} :_T f) \subseteq \mathfrak{b}$, or equivalently, $(\mathfrak{d} :_T \mathfrak{b}) \subseteq (\mathfrak{d}, f) \subseteq (\mathfrak{d} :_T \mathfrak{c})$.

Setup: For the rest of this section, we will work modulo $\mathfrak{d}$, and use the same notation to denote $\mathfrak{b}$, $\mathfrak{c}$ and f modulo $\mathfrak{d}$.

Let $\mathfrak{B} = \{m_1, \ldots, m_r\}$ be a monomial k-basis for $R := T/\mathfrak{d}$. Write $f = \sum_{i=1}^{r} Z_i m_i$, where Z_i are indeterminates over k. Let $Z = (z_{ij})_{r \times r}$ be matrix representing multiplication by f on R, with respect to the basis $\mathfrak{B}$.

Proposition 1.5. *The set of all $f \in R$ such that $\mathfrak{c} \subseteq (0 :_R f) \subseteq \mathfrak{b}$ corresponds to a closed set in k^{r+rt} (in the Zariski topology).*

Proof. Let m_i, f and Z be as in the above setup. Write $\mathfrak{c} = (\sum_j b_{ij}m_j : 1 \leq i \leq s)$ and $(0 :_R \mathfrak{b}) = (\sum_j a_{ij}m_j : 1 \leq i \leq t)$. Let $B = (b_{ij})_{r\times s}$, $A = (a_{ij})_{r\times t}$ and $Y = (Y_{ij})_{t\times r}$ be a matrix of indeterminates over k.

So there is an $f \in R$ satisfying

(i)$(0 :_R \mathfrak{b}) \subseteq (f)$ if and only if the system of equations $A = YZ$ has a solution over k and

(ii) $\mathfrak{c} \subseteq (0 :_R f)$ if and only if the system of equations $BZ = 0$ has a solution over k.

Thus the set of all $f \in R$ such that $\mathfrak{c} \subseteq (0 :_R f) \subseteq \mathfrak{b}$ corresponds to a closed set in k^{r+rt}; the set of solutions to the system of equations $A = YZ$ and $BZ = 0$. □

Lemma 1.6. *With notations as in the setup above,* $\lambda(R/(0 :_R f)) = \operatorname{rank}(Z)$.

Proof. The exact sequence

$$0 \longrightarrow (0 :_R f) \longrightarrow R \xrightarrow{\cdot f} R \longrightarrow R/(f) \longrightarrow 0$$

gives us the equality $\lambda(0 :_R f) = \lambda(R/(f))$. (Another way to look at this equality is that $0 :_R f$ is isomorphic to the canonical module of $R/(f)$). Since $R/(f)$ is the cokernel of multiplication by f on R, $\lambda(R/(f)) = \lambda(R) - \operatorname{rank}(Z)$. Since

$$\lambda(R) - \lambda(R/(f)) = \lambda(R) - \lambda(0 :_R f) = \lambda(R/(0 :_R f)),$$

we get the required equality. □

Definition 1.7. *We say that a set X is constructible if $X = \cap_{i=1}^{n} X_i$ where each X_i is the intersection of an open and a closed set.*

We use the following notation for the next theorem:

Let $\mathfrak{b}_j$ be the ideal of the $j \times j$ minors of Z in $\mathsf{k}[Z_1, \ldots, Z_r, Y_{ij}]$. Let V_j be the set of points in k^{r+rt} which are the common zeroes of the polynomials in $\mathfrak{b}_j$. Then V_j is closed in the Zariski topology.

Theorem 1.8. *Given a positive integer n, $\lambda(R/\mathfrak{b}) \leq n \leq \lambda(R/\mathfrak{c})$, the set of all $f \in R$ such that $\mathfrak{c} \subseteq (0:_R f) \subseteq \mathfrak{b}$ and $\lambda(R/(0:_R f)) = n$ corresponds to a consrtuctible set in* k^{r+rt}.

Proof. By Proposition 1.5, the existence of an f such that $\mathfrak{c} \subseteq (0:_R f) \subseteq \mathfrak{b}$ corresponds to a closed set, say V in k^{r+rt}. By the previous lemma, if $\lambda(R/(0:_R f)) = n$, then f satisfies $\lambda(R/(0:_R f)) = n$ if and only if $\operatorname{rank}(Z) = n$. Thus such an f corresponds to a solution of the system of equations obtained by setting all $i \times i$ minors of Z (for $i > n$) equal to 0, which does not satisfy at least one of the equations obtained by setting the $n \times n$ minors of Z equal to 0. Therefore, an f satisfying both conditions corresponds to a point in a constructible set in k^{r+rt}; an intersection section of closed sets (V_{n+1} and V) and an open set (the complement of V_n). □

1.4 A Different(ial) Point of View

Let $T = \mathsf{k}[X_1, \ldots, X_d]$ be a polynomial ring over k. One can see, using local cohomology and local duality, that the injective hull of k over T is $E' = \mathsf{k}[X_1^{-1}, \ldots, X_d^{-1}]$, where the multiplication is defined by

$$(X_1^{a_1} \cdots X_d^{a_d}) \cdot (X_1^{-b_1} \cdots X_d^{-b_d}) = \begin{cases} X_1^{a_1-b_1} \cdots X_d^{a_d-b_d} & \text{if } a_i \leq b_i \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

and extended linearly (e.g., see [17]).

In other words, if m is a monomial in T, n is a monomial (in X_i^{-1}) in E', then $m \cdot n$ is defined the usual way if it is in E', else the product is defined to be zero.

For the further results in this section, we will assume that $\operatorname{char}(\mathsf{k}) = 0$. Consider the T-module $E = \mathsf{k}[Y_1, \ldots, Y_d]$, where the X_i's act on the Y_j's by partial differentiation, i.e.

$$f(X_1, \ldots, X_d) \cdot g(Y_1, \ldots, Y_d) = f\left(\frac{\partial}{\partial Y_1}, \ldots, \frac{\partial}{\partial Y_d}\right)[g(Y_1, \ldots, Y_d)].$$

Proposition 1.9. *Let $T = \mathsf{k}[X_1, \ldots, X_d]$, $E' = \mathsf{k}[X_1^{-1}, \ldots, X_d^{-1}]$ and $E = \mathsf{k}[Y_1, \ldots, Y_d]$ be T-modules with multiplication defined as above, where k is a field of characteristic zero. Then $E \simeq E'$ as a T-module, i.e., E is isomorphic to the injective hull of k over T.*

Proof. Consider the map $\Phi : E \longrightarrow E'$ defined by

$$Y_1^{b_1} \cdots Y_d^{b_d} \overset{\Phi}{\mapsto} (b_1!) \cdots (b_d!) X_1^{-b_1} \cdots X_d^{-b_d},$$

and extended linearly. We claim that Φ is an R-module isomorphism.

Clearly, by considering Φ to be a map of k-vector spaces, Φ is bijective. Thus we need to only prove that Φ is an R-module homomorphism.

Since the multiplication on both the modules is defined on monomials and extended linearly, to see that Φ is an R-module homomorphism, it is enough to consider the action of monomials in R on monomials in E and E'. Thus it is enough to show that

$$\Phi((X_1^{a_1} \cdots X_d^{a_d}) \cdot (Y_1^{b_1} \cdots Y_d^{b_d})) = (X_1^{a_1} \cdots X_d^{a_d}) \cdot \Phi(Y_1^{b_1} \cdots Y_d^{b_d}),$$

$$\text{i.e., } \Phi\left(\frac{\partial^{a_1}}{\partial Y_1^{a_1}} \cdots \frac{\partial^{a_d}}{\partial Y_d^{a_d}}(Y_1^{b_1} \cdots Y_d^{b_d})\right) = (X_1^{a_1} \cdots X_d^{a_d}) \cdot \left[(b_1!) \ldots (b_d!) X_1^{-b_1} \cdots X_d^{-b_d}\right].$$

If $a_i > b_i$ for some i, then both sides are zero and hence equal. Now consider the case when $a_i \leq b_i$ for all i.

The left hand side is

$$\Phi\left(\tfrac{b_1!}{(b_1-a_1)!}\cdots\tfrac{b_d!}{(b_d-a_d)!}\,Y_1^{b_1-a_1}\cdots Y_d^{b_d-a_d}\right)$$

$$=\tfrac{b_1!}{(b_1-a_1)!}\cdots\tfrac{b_d!}{(b_d-a_d)!}\,(b_1-a_1)!\cdots(b_d-a_d)!\,X_1^{a_1-b_1}\cdots X_d^{a_d-b_d}$$

$$=(b_1!)\dots(b_d!)\,X_1^{a_1-b_1}\cdots X_d^{a_d-b_d}$$

which is equal to the right hand side. □

Remark 1.10. Let $\mathfrak{m}=(X_1,\cdots,X_d)$ be the unique homogeneous maximal ideal and $\mathfrak{b}$ an $\mathfrak{m}$-primary ideal in T. Then the quotient ring $R:=T/\mathfrak{b}$ is an Artinian local ring. Since R is Artinian, the injective hull of $\Bbbk$ over R is isomorphic to the canonical module ω_R of R. Hence

$$\omega_R\simeq \mathrm{Hom}_T(R,E)\simeq \mathrm{Hom}_T(T/\mathfrak{b},E)\simeq(0:_E\mathfrak{b}).$$

Thus, ω_R is isomorphic to a finitely generated submodule of E.

On the other hand, consider finitely many polynomials $g_1,\cdots,g_r$ in E. Then $\mathfrak{b}=(0:_T(g_1,\cdots,g_r))$ is the collection of partial derivatives which annihilate all the g_j's. Since there are only finitely many g_j's, $(\partial_i^{a_i}/\partial Y_i^{a_i})(g_j)=0$ for some a_i large enough. Hence there are positive integers a_i such that $X_i^{a_i}\in\mathfrak{b}$, i.e., $\mathfrak{b}$ is $\mathfrak{m}$-primary. Let $R:=T/\mathfrak{b}$. Therefore, by duality,

$$(g_1,\dots,g_r)=(0:_E(0:_T(g_1,\dots,g_r)))=(0:_E\mathfrak{b})\simeq\omega_R.$$

Thus there is a one-one correspondence between finitely generated T-submodules of E and $\mathfrak{m}$-primary ideals in T.

Remark 1.11. What are the ideals corresponding to submodules of E generated by a single element? Let $\mathfrak{c}$ be an ideal such that $\mathfrak{c} = (0 :_T g)$ for some $g \in E$. Then $S := T/\mathfrak{c}$ is an Artinian ring with canonical module $\omega_S \simeq (g)$, i.e., ω_S is cyclic. This is equivalent to S being a Gorenstein ring. Thus $T/\mathfrak{c}$ is a Gorenstein Artin local ring if and only if $\mathfrak{c} = (0 :_T g)$ for some $g \in E$.

The problem we are interested in is:

Question 1.12. Given an $\mathfrak{m}$-primary ideal $\mathfrak{b}$ in T, can we find an $\mathfrak{m}$-primary Gorenstein ideal $\mathfrak{c}$ (i.e., $T/\mathfrak{c}$ is a Gorenstein ring) contained in $\mathfrak{b}$ such that $\lambda(\mathfrak{b}/\mathfrak{c})$ is minimum?

Let $(0 :_E \mathfrak{b}) = (g_1, \ldots, g_r)$. If $\mathfrak{c} \subseteq \mathfrak{b}$ is a Gorenstein ideal, then $(0 :_E \mathfrak{b}) \subseteq (0 :_E \mathfrak{c}) = (g)$ for some $g \in E$. Thus the above problem can be rephrased in terms of the annihilators of $\mathfrak{b}$ and $\mathfrak{c}$ in E as:

Question 1.13. Given polynomials $g_1, \ldots, g_r$, can we find a single polynomial g such that each g_j is some partial derivative of g and $\lambda((0 :_T (g_1, \ldots, g_r))/(0 :_T g))$ is minimum?

Idealization and the Corresponding Polynomial

Consider the polynomial $g = Z_1 g_1 + \cdots + Z_r g_r \in E[Z_1, \ldots, Z_r]$ where $E[Z_1, \ldots, Z_r]$ is a module over $T[U_1, \ldots, U_r]$. The U_i's act on the Z_j's by partial differentiation, i.e., $U_i \cdot Z_j = \delta_{ij}$. Note that $U_i \cdot g = g_i$. The question one asks is: Which Gorenstein Artin ring does this g correspond to?

We claim that this corresponds to the idealization $R \ltimes \omega_R$ we have described in Section 2. Note that $R \ltimes \omega_R \simeq T[U_1, \ldots, U_r]/\mathfrak{c}$, where

$$\mathfrak{c} = \mathfrak{b} + (U_1, \ldots, U_r)^2 + \left(\sum_{i=1}^{r} t_i U_i : t_i \in T \text{ are such that } \sum_{i=1}^{r} (t_i \cdot g_i) = 0 \right).$$

Thus, to prove that g corresponds to $R \ltimes \omega_R$, we need to prove that $\mathfrak{c} = (0 :_{T[U_1, \ldots, U_r]} g)$.

Clearly $\mathfrak{b} + (U_1, \ldots, U_r)^2 \subseteq (0 :_{T[U_1,\ldots,U_r]} g)$, since $\mathfrak{b} = (0 :_T (g_1, \ldots, g_r))$ and g is linear in the Z_i's.

Now, note that $(\sum_{i=1}^r t_i U_i) \cdot g = \sum_{i=1}^r (t_i \cdot g_i)$. Hence if $\sum_{i=1}^r (t_i \cdot g_i) = 0$, then $\sum_{i=1}^r (t_i U_i) \in (0 :_{T[U_1,\ldots,U_r]} g)$. Thus $\mathfrak{c} \subseteq (0 :_{T[U_1,\ldots,U_r]} g)$.

To prove the converse, let $f \in (0 :_{T[U_1,\ldots,U_r]} g)$. We can write $f = f_x + f_u + \sum_{i=1}^r (t_i U_i)$, where $f_x \in (X_1, \ldots, X_r)$, $f_u \in (U_1, \ldots, U_r)^2$ and $t_i \in T$. Since $f \cdot g = 0$, we have $(f_x + f_u + \sum_{i=1}^r (t_i U_i)) \cdot g = 0$. Note that $f_u \cdot g = 0$. Hence $f_x \cdot g + \sum_{i=1}^r (t_i \cdot g_i) = 0$. Observe that every term of $f_x \cdot g = \sum_{i=1}^r Z_i (f_x \cdot g_i)$ is homogeneous of degree 1 in the Z_i's, whereas $\sum_{i=1}^r (t_i \cdot g_i)$ is purely in terms of the Y_i's. Hence $f_x \cdot g_i = 0$ for each i, and $\sum_{i=1}^r (t_i \cdot g_i) = 0$. Thus $f_x \in (0 :_T (g_1, \ldots, g_r)) = \mathfrak{b}$, $f_u \in (U_1, \ldots, U_r)^2$ and $\sum_{i=1}^r (t_i U_i) \in (\sum_{i=1}^r (t_i U_i) : t_i \in T$ are such that $\sum_{i=1}^r (t_i \cdot g_i) = 0)$. Thus $f \in \mathfrak{c}$, proving the claim.

Chapter 2

The Gorenstein Colength of an Artinian Local Ring

2.1 Gorenstein Colength: Basic Properties

We will use the following notation in this chapter.

Setup 2.1.
1. Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring and ω_R (or simply ω) be the canonical module of R. Note that since R is Artinian, ω is the same as E, the injective hull over R, of the residue field k.
By $(\_)^*$ and $(\_)^\vee$, we mean $\mathrm{Hom}_R(\_, R)$ and $\mathrm{Hom}_R(\_, \omega)$ respectively.

2. By Cohen's Structure Theorem, we can write R as the quotient of a regular local ring $(T, \mathfrak{m}_T, \mathsf{k})$. Let $\mathfrak{b}$ be an ideal in T such that $R \simeq T/\mathfrak{b}$.
By $\bar{\ }$, we mean going modulo $\mathfrak{b}$.

The question we are interested in is the following:
Given an Artinian local ring $(R, \mathfrak{m}_R, \mathsf{k})$, how "close" can one get to R by a Gorenstein Artin local ring? This can be made precise using the following definition:

Definition 2.2. *Let $(R, \mathfrak{m}_R, \mathsf{k})$ be an Artinian local ring. Define the Gorenstein colength of R as:*

$$g(R) = \min\{\lambda(S) - \lambda(R) : S \text{ is a Gorenstein Artin local ring mapping onto } R\},$$

where $\lambda(\_)$ denotes length.

The number $g(R)$ gives a numerical value to how close one can get to an Artinian local ring R by a Gorenstein Artin local ring.

Remark 2.3. We do not require that the embedding dimension of S be the same as that of R.

Remark 2.4.

1. Note that $g(R)$ is zero if and only if R is Gorenstein.
2. When is $g(R) = 1$? Recall that if S is a Gorenstein Artin local ring, then $\mathrm{soc}(S)$ has length 1 and is contained in every ideal of S. Thus, $g(R) = 1$ if and only if R is not Gorenstein and $R \simeq S/\mathrm{soc}(S)$ for a Gorenstein Artin ring S, i.e. R is a Teter ring. The Huneke-Vraciu Theorem (Theorem 0.3) and Teter's Theorem (Theorem 0.2) give a characterization for such rings.

Proposition 2.5. *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring. Then $g(R)$ is finite.*

Proof. As in Setup 2.1(2), by Cohen's Structure Theorem we can write $R \simeq T/\mathfrak{b}$ where $(T, \mathfrak{m}_T, \mathsf{k})$ is a regular local ring and $\mathfrak{b}$ is an $\mathfrak{m}_T$-primary ideal. If $\dim(T) = d$, choose a regular sequence $x_1, \ldots, x_d$ in $\mathfrak{b}$ and set $S := T/(x_1, \ldots, x_d)$. If we set $\bar{\mathfrak{b}} := \mathfrak{b}/(x_1, \ldots, x_d)$, we see that $R \simeq S/\bar{\mathfrak{b}}$. Then S is a complete intersection ring and hence a Gorenstein Artin local ring mapping onto R. Thus $g(R) \leq \lambda(S) - \lambda(R)$ which is finite. $\square$

Remark 2.6. If k is infinite, then it is known (e.g., see [21]) that since $\mathfrak{b}$ is $\mathfrak{m}_T$-primary, any minimal reduction of $\mathfrak{b}$ is generated by d elements and hence is a complete intersection. Thus the proof above shows that if k is infinite, then by choosing $\mathfrak{c}$ to be a minimal reduction of $\mathfrak{b}$,

$$g(R) \leq e_0(\mathfrak{b}) - \lambda(R),$$

where $e_0(\mathfrak{b}) = \lambda(S)$ is the multiplicity of $\mathfrak{b}$.

Proposition 2.7. *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring. Then $g(R) \leq \lambda(R)$.*

Proof. Let ω be the canonical module of R and $S = R \ltimes \omega$ be the idealization of the canonical module. Then S is a Gorenstein Artin local ring mapping onto R.

Since $\lambda(S) = 2\lambda(R)$, and S maps onto R via the natural projection, $g(R) \leq \lambda(S) - \lambda(R) = \lambda(R)$. $\square$

Example 2.8. In this example, we see that $g(R) < \min\{e_0(\mathfrak{b}) - \lambda(R), \lambda(R)\}$ with notation as in Remark 2.6.

Let $T = \mathbb{Q}[X,Y,Z]$, $\mathfrak{b} = (X^2, XY, XZ, Y^2, YZ, Z^2)$ and $R := T/\mathfrak{b}$. Note that $e_0(\mathfrak{b}) = 8$, $\lambda(R) = 4$.

Let $\mathfrak{c} = (X^2 - Y^2, X^2 - Z^2, XY, XZ, YZ)$ and $S := T/\mathfrak{c}$. Then S is a Gorenstein Artin ring and maps onto R since $\mathfrak{c} \subseteq \mathfrak{b}$. Since $\lambda(S) = 5$ and R is not Gorenstein, we see that $g(R) = 1$.

2.2 Self-dual Ideals and the Fundamental Inequalities

Definition 2.9. *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring and ω be the canonical module of R. We say that an ideal $\mathfrak{a} \subseteq R$ is self-dual if $\mathfrak{a} \simeq \mathfrak{a}^\vee$ $(= \mathrm{Hom}_R(\mathfrak{a}, \omega))$.*

Remark 2.10.

1. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are self-dual ideals such that $\mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$, then $\mathfrak{a}_1 + \mathfrak{a}_2$ is also self-dual, since $\mathfrak{a}_1 + \mathfrak{a}_2 \simeq \mathfrak{a}_1 \oplus \mathfrak{a}_2$

2. In particular, if $\mathfrak{a}$ is self-dual, then so is $\mathfrak{a} + \mathrm{soc}(R)$.

As one can see from the Huneke-Vraciu Theorem (Theorem 0.3) and Teter's Theorem (Theorem 0.2), Gorenstein colength is closely related to self-dual ideals.

Definition 2.11. *We say that the map $f : \omega \longrightarrow \mathfrak{a}$ (resp. $\phi : \mathfrak{a} \longrightarrow \mathfrak{a}^\vee$) satisfies Teter's condition if the commutativity condition $f(x)y = f(y)x$ for all $x,y \in \omega$ (resp. $\phi(x)(y) = \phi(y)(x)$ for all x, $y \in \mathfrak{a}$) is satisfied.*

Remark 2.12. Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring and $\mathfrak{a}$ be an ideal in R. The inclusion map $\mathfrak{a} \overset{i}{\hookrightarrow} R$ induces a surjective map $\omega \simeq \mathrm{Hom}(R, \omega) \xrightarrow{i^\vee} \mathrm{Hom}_R(\mathfrak{a}, \omega)$ which is given by $u \mapsto u \circ i$. Thus we see that for every $a \in \mathfrak{a}$ and $u \in \omega$, $i^\vee(u)(a) = au$.

Lemma 2.13. *Let $\mathfrak{a}$ be an ideal in R. The following are equivalent:*

a) *There is an isomorphism $\phi : \mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^\vee$.*

b) *There is a surjective map $\omega \overset{f}{\twoheadrightarrow} \mathfrak{a}$ such that $\ker(f) = 0 :_\omega \mathfrak{a}$.*

Moreover $\phi(r)(s) = \phi(s)(r)$ for all r, $s \in \mathfrak{a}$ if and only if $f(x)y = f(y)x$ for all x, $y \in \omega$.

Remark 2.14. Note that the last part of the lemma says that $\phi : \mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^\vee$ satisfies Teter's condition if and only if $f : \omega \twoheadrightarrow \mathfrak{a}$ satisfies Teter's condition.

Proof.

(a) $\Rightarrow$ (b): Consider the short exact sequence $0 \to \mathfrak{a} \xrightarrow{i} R \to R/\mathfrak{a} \to 0$. Applying $\mathrm{Hom}(\_, \omega)$, we get the short exact sequence $0 \to 0 :_\omega \mathfrak{a} \to \omega \xrightarrow{i^\vee} \mathfrak{a}^\vee \to 0$.

Let $f = \phi^{-1} \circ i^\vee : \omega \twoheadrightarrow \mathfrak{a}$. Since ϕ is an isomorphism, $\ker(f) = \ker(i^\vee) = 0 :_\omega \mathfrak{a}$.

Now suppose ϕ satisfies Teter's condition. Let u, $v \in \omega$. Then $\phi(f(u))(f(v)) = \phi(f(v))(f(u))$. Since $\phi \circ f = i^\vee$, we have $i^\vee(u)(f(v)) = i^\vee(v)(f(u))$. By Remark 2.12, this shows that $f(v)u = f(u)v$, i.e., f satisfies Teter's condition.

(b) $\Rightarrow$ (a): The proof follows from the following diagram; note that the maps $0:_\omega \mathfrak{a} \hookrightarrow \omega$ and $\ker(f) \hookrightarrow \omega$ are the natural inclusions.

$$\begin{tikzcd}
0 \arrow[r] & 0:_\omega \mathfrak{a} \arrow[r] \arrow[d, equal] & \omega \arrow[r, "i^\vee"] \arrow[d, equal] & \mathfrak{a}^\vee \arrow[r] & 0 \\
0 \arrow[r] & \ker(f) \arrow[r] & \omega \arrow[r, "f"] & \mathfrak{a} \arrow[r] \arrow[u, dashed, "\phi", "\simeq"'] & 0
\end{tikzcd}$$

Let us now prove that $\phi(x)(y) = \phi(y)(x)$ for all x, $y \in \mathfrak{a}$, assuming that f satisfies Teter's condition. Let u, $v \in \omega$ be such that $f(u) = x$ and $f(v) = y$.

$$\begin{aligned}
\phi(x)(y) &= \phi(f(u))(f(v)) \\
&= i^\vee(u)(f(v)) && \text{since } \phi \circ f = i^\vee \\
&= f(v)u && \text{by Remark 2.12} \\
&= f(u)v && \text{since } f \text{ satisfies Teter's condition} \\
&= i^\vee(v)(f(u)) \\
&= \phi(f(v))(f(u)) \\
&= \phi(y)(x),
\end{aligned}$$

i.e., ϕ satisfies Teter's condition. $\square$

Let us now see what happens when a Gorenstein Artin local ring S maps onto the given Artinian local ring R. We summarize our observations in the next proposition. These lead to lower bounds on $g(R)$.

Proposition 2.15. *Let $(S, \mathfrak{m}_S, \mathsf{k})$ be a Gorenstein Artin local ring and $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring with canonical module ω. Let $\psi : S \longrightarrow R$ be a surjective ring* *homomorphism such that* $\ker(\psi) = \mathfrak{b}$. *Then*

1) ω *is isomorphic to an ideal* W *in* S.

2) *Identifying* ω *with* W, $\ker(f) \cdot f(\omega) = 0$ *where* $f = \psi|_{\omega}$ *and*

3) $f : \omega \longrightarrow R$ *satisfies Teter's condition.*

Proof.

1) S is a Gorenstein ring of the same dimension mapping onto R. Therefore $\omega \simeq \operatorname{Hom}_S(R,S) \simeq (0 :_S \mathfrak{b}) \subseteq S$.

2) We have $\omega \simeq (0 :_S \mathfrak{b})$, $f(\omega) \simeq ((0 :_S \mathfrak{b}) + \mathfrak{b})/\mathfrak{b}$ and $\ker(f) \simeq \mathfrak{b} \cap (0 :_S \mathfrak{b})$. Hence $\ker(f) \cdot f(\omega) = 0$.

3) Since the elements of ω can be identified with elements of S, for any x, y in ω, $f(x)y = f(y)x$. □

Corollary 2.16. *With notations as in Proposition 2.15, the ideal* $\mathfrak{a} := f(\omega)$ *is self-dual.*

Proof. In order to prove that $\mathfrak{a}$ is self-dual, by Lemma 2.13, we only need to show that $\ker(f) = 0 :_{\omega} \mathfrak{a}$.

Note that $0 :_{\omega} \mathfrak{a} \simeq (R/\mathfrak{a})^{\vee}$, hence

$$\begin{aligned} \lambda(0 :_{\omega} \mathfrak{a}) &= \lambda(R/\mathfrak{a}) \\ &= \lambda(\omega) - \lambda(\mathfrak{a}) \\ &= \lambda(\ker(f)) \end{aligned}$$

since $\omega/\ker(f) \simeq \mathfrak{a}$. Now, by Proposition 2.15(2), we know that $\ker(f) \subseteq 0 :_{\omega} \mathfrak{a}$. Thus $\ker(f) = 0 :_{\omega} \mathfrak{a}$ since they have the same length. □

Lemma 2.17. *With notation as in Proposition 2.15,* $\lambda(S) - \lambda(R) \geq \lambda(R/\psi(\omega))$. *Moreover equality holds, i.e.,* $\lambda(S) - \lambda(R) = \lambda(R/\psi(\omega))$ *if and only if* $\mathfrak{b}^2 = 0$.

Proof. Let $\mathfrak{a} = \psi(\omega)$. The isomorphism $\omega \simeq (0 :_S \mathfrak{b})$ in S yields $\lambda(R) = \lambda((0 :_S \mathfrak{b})$. Since $S/(\omega + \mathfrak{b}) \simeq R/\mathfrak{a}$, the first statement of the lemma is proved if we show $\lambda(S) - \lambda(0 :_S \mathfrak{b}) \geq \lambda(R/\mathfrak{a})$, i.e., if $\lambda(S/(0 :_S \mathfrak{b})) \geq \lambda(S/((0 :_S \mathfrak{b}) + \mathfrak{b}))$.

But this is always true. Moreover, equality holds, i.e., $\lambda(S) - \lambda(R) = \lambda(\omega) - \lambda(\mathfrak{a})$ if and only if $\mathfrak{b} \subseteq (0 :_S \mathfrak{b})$, i.e., $\mathfrak{b}^2 = 0$. $\square$

The following is a useful consequence of the above lemma, which gives us a lower bound on $g(R)$.

Corollary 2.18. *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring with canonical module ω. Then $g(R) \geq \min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \subseteq R$ is a self-dual ideal$\}$. In particular, $g(R) \geq \lambda(R/(\omega^*(\omega)))$.*

Proof. Let S be any Gorenstein Artin local ring and $\psi : S \twoheadrightarrow R$ be a surjective ring homomorphism. By Lemma 2.17, $\lambda(S) - \lambda(R) \geq \lambda(R/\mathfrak{a})$ and by Corollary 2.16, $\mathfrak{a}$ is a self-dual ideal. Thus $g(R) \geq \min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\}$.

The last statement in the corollary follows from Lemma 2.13, since $\mathfrak{a} \subseteq \omega^*(\omega)$ for every self-dual ideal $\mathfrak{a}$. $\square$

Thus, with notation as before, we see that

$$\boxed{\lambda(R/(\omega^*(\omega))) \leq \min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\} \leq g(R) \leq \lambda(R).}$$

Fundamental Inequalities

A natural question at this juncture is the following:

Question 2.19. Is $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\} = g(R)$?

A stronger question one can ask is:

Question 2.20. Given a self-dual ideal $\mathfrak{a}$ in R, is there a Gorenstein Artin local ring S such that $\lambda(S) - \lambda(R) = \lambda(R/\mathfrak{a})$?

We answer Question 2.20 in a special case in Theorem 2.32. The machinery we need to prove the theorem is developed in the next section.

2.3 The Dual of the Canonical Module

Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring with canonical module ω. In this chapter, we record some of the results relating to the dual $\omega^* = \mathrm{Hom}_R(\omega, R)$ of ω.

Computing $\omega^*(\omega)$

Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring with canonical module ω. We will see how one can compute the trace ideal $\omega^*(\omega)$ of ω.

Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring mapping onto R. Let

$$0 \to T^{b_d} \xrightarrow{\phi} T^{b_{d-1}} \to \ldots \to T \to R \to 0$$

be a minimal resolution of R over T. Then a resolution of the canonical module ω of R over T is given by taking the dual of the above resolution, i.e., by applying $\mathrm{Hom}_T(\_, T)$ to the above resolution. Hence a presentation of ω is $T^{b_{d-1}} \xrightarrow{\phi^*} T^{b_d} \to \omega \to 0$. Tensor with R and apply $\mathrm{Hom}_R(-, R)$ to get an exact sequence $R^{b_{d+1}} \xrightarrow{\psi} R^{b_d} \xrightarrow{\phi \otimes R} R^{b_{d-1}}$, where $\omega^* = \ker(\phi \otimes R) = \mathrm{Im}(\psi)$.

Lemma 2.21. *With notation as above, let ψ be given by the matrix (a_{ij}). Then the trace ideal of ω, $\omega^*(\omega)$, is the ideal generated by the a_{ij}'s.*

The above lemma is a particular case of the following lemma.

Lemma 2.22. *Let $(R,\mathfrak{m},\mathsf{k})$ be a Noetherian local ring and M a finitely generated R-module. Let $R^n \xrightarrow{B} R^m \longrightarrow M \longrightarrow 0$ be a minimal presentation of M. Apply* $\operatorname{Hom}_R(\_,R)$ *to get an exact sequence $0 \longrightarrow M^* \longrightarrow (R^*)^m \xrightarrow{B^*} (R^*)^n$. Map a free R-module, say R^k, minimally onto M^* to get an exact sequence $R^k \xrightarrow{A} (R^*)^m \xrightarrow{B^*} (R^*)^n$, where $M^* = \ker(B^*) = \operatorname{Im}(A)$. Then the trace ideal of M, $M^*(M) = (a_{ij}$: a_{ij} are the entries of the matrix $A)$.*

Proof. Let $m_1,\ldots,m_n$ be a minimal generating set of M, $e_1,\ldots,e_m$ be a basis of R^m such that $e_i \mapsto m_i$, and $e_1^*,\ldots,e_m^*$ be the corresponding dual basis of $(R^*)^m$.

Let $f \in M^*$. Write $f = \Sigma_{i=1}^m r_i e_i^* \in (R^*)^m$. Then f acts on M by sending m_j to r_j. Hence if $A = (a_{ij})$, then the generators of M^* are $f_j = \Sigma_{i=1}^m a_{ij} e_i^*$, $1 \le j \le k$. Thus $f_j(m_i) = a_{ij}$. Thus $M^*(M) = (a_{ij})$. $\square$

Remark 2.23. Let R be a quotient of a polynomial ring $T = \mathsf{k}[X_1,\ldots,X_d]$ by an ideal I, which is primary to $(X_1,\ldots,X_d)$. One can use the above lemma to compute the trace ideal of the canonical module ω of R in Macaulay 2. For example, if $R \simeq T/I$, where T = k[X,Y,Z], then the following command can be used: ker(res(T^1/I).dd_3 ** T/I). The output is the matrix A whose entries give the trace ideal of ω.

An Involution on ω^*

Remark 2.24. Let U, V and W be R-modules. Consider the series of natural isomorphisms

$$\operatorname{Hom}(U,\operatorname{Hom}(V,W)) \simeq \operatorname{Hom}(U\otimes V,W)$$

$$\simeq \operatorname{Hom}(V\otimes U,W) \simeq \operatorname{Hom}(V,\operatorname{Hom}(U,W)).$$

Let $f^* \in \operatorname{Hom}(V,\operatorname{Hom}(U,W))$ be the image of a map $f \in \operatorname{Hom}(U,\operatorname{Hom}(V,W))$ under the series of isomorphisms. Then $f(u)(v) = f^*(v)(u)$ for all $u \in U$ and $v \in V$.

Thus if $U = V$, we get an involution on $\mathrm{Hom}(U, \mathrm{Hom}(U,W))$ induced by the involution $u \otimes v \mapsto v \otimes u$ on $U \otimes U$. In this case, $(f^*)^* = f$.

In their paper [12], Huneke and Vraciu construct an involution adj on ω^* as follows:

Let $f \in \mathrm{Hom}_R(\omega, R)$. Fix $u \in \omega$. Consider $\phi_{f,u} : \omega \to \omega$ defined by $\phi_{f,u}(v) = f(v) \cdot u$. Since $\mathrm{Hom}_R(\omega, \omega) \simeq R$, there is an element $r_{f,u} \in R$ such that $\phi_{f,u}(v) = r_{f,u} \cdot v$.

Define $f^* : \omega \to R$ by $f^*(u) = r_{f,u}$. We can now define $adj : \omega^* \longrightarrow \omega^*$ as $adj(f) = f^*$. One can see that $f^* \in \omega^*$ and that $f^*(u)(v) = f(v)(u)$ for all u, $v \in \omega$. Moreover adj is an involution on ω^* since $(f^*)^* = f$.

This involution is the same as the one described in Remark 2.24, with $U = V = W = \omega$. Note that in this case, $\mathrm{Hom}(\omega, \mathrm{Hom}(\omega, \omega)) \simeq \omega^*$.

The following remarks follow immediately from the definition of adj.

Remark 2.25.

1) $\ker(f) = (0 :_\omega f^*(\omega))$; $f^*(\omega) = (0 :_R \ker(f))$ and vice versa.

2) Since ω is a faithful R-module, we see that $f = f^*$ if and only if $f(x)y = f(y)x$ for all x, $y \in \omega$, i.e., f satisfies Teter's condition. Thus it follows from (1) that when $f = f^*$, $\ker(f) = 0 :_\omega f(\omega)$ and $f(\omega) = 0 :_R \ker(f)$, i.e., $f(\omega)$ is a self-dual ideal in R.

3) As in the proof of Corollary 2.16, $\lambda(\ker(f)) = \lambda(R/f(\omega)) = \lambda(0 :_\omega f(\omega))$ by duality. Therefore, if $f(\omega) \cdot \ker(f) = 0$, then $\ker(f) = (0 :_\omega f(\omega)) = \ker(f^*)$. Thus we see that $\ker(f) \cdot f(\omega) = 0$ if and only if $\ker(f) = (0 :_\omega f(\omega)) = \ker(f^*)$ if and only if $f(\omega) = f^*(\omega)$.

In particular, the above equivalent conditions follow from the commutativity condition $f(x)y = f(y)x$ for all x, $y \in \omega$ (or equivalently $f = f^*$).

Remark 2.26. Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring. Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal in R such that there is a surjective map $f : M \twoheadrightarrow \mathfrak{a}$ with $\mathfrak{a}(\ker f) = 0$. Since $f(x)y - f(y)x \in \ker(f)$, for any $w \in M$, $f(w)[f(x)y - f(y)x] = 0$.

Thus

$$f(w)f(x)y = f(w)f(y)x \text{ for all } w,x,y \in M. \qquad (\sharp)$$

The following proposition plays a key role in our proof of Theorem 2.32 and in Corollary 2.36 of our main Theorem 2.34. In this proposition, we use an operator *sym* on ω^* corresponding to the involution *adj* which can be defined as follows:

Given a map $f \in \omega^*$, we can define $sym(f) = f + adj(f)$, i.e. $sym(f) = f + f^*$. Then $adj(sym(f)) = sym(f)$, i.e., for any $f \in \omega^*$, $sym(f)$ satisfies Teter's condition.

Proposition 2.27. *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring with canonical module ω. Let $f \in \omega^*$ be such that $\ker(f) = (0 :_\omega \mathfrak{a})$ where $\mathfrak{a} := f(\omega)$. Assume that 2 is invertible in R. Then there is a map $h : \omega \longrightarrow R$ satisfying:*

1) *$h(x)y = h(y)x$ for all x, $y \in \omega$, i.e., h satisfies Teter's condition.*

2) $\ker(h) \cap \mathfrak{a} \cdot \omega \subseteq \ker(f)$.

3) $\ker(f) \subseteq \ker(h) \subseteq (0 :_\omega \mathfrak{a}^2)$, *i.e.,* $\mathfrak{a}^2 \subseteq h(\omega) \subseteq \mathfrak{a}$.

4) *If* $(0 :_R \mathfrak{a}) \subseteq \mathfrak{a}^2$, *then* $\ker(f) = \ker(h)$ (*or equivalently* $h(\omega) = \mathfrak{a}$).

Proof. Define $h = sym(f) = f + f^*$. Then $h = h^*$, i.e., h satisfies Teter's condition. By Remark 2.25(3), this implies that $\ker(h) = (0 :_\omega h(\omega))$.

We see that by definition of h, $\ker(f) \cap \ker(f^*) \subseteq \ker(h)$. But by Remark 2.25(3) (and the assumption that $\ker(f) \cdot f(\omega) = 0$), $\ker(f) = \ker(f^*)$. Hence $\ker(f) \subseteq \ker(h)$ giving the first inclusion in (3). The other inclusion in (3) is a consequence of (1) and (2) which can be seen as follows: By (2), $\mathfrak{a} \cdot \ker(h) \subseteq \ker(f)$. Thus $\ker(h) \subseteq (\ker(f) :_\omega \mathfrak{a})$ which gives us $\ker(h) \subseteq (0 :_\omega \mathfrak{a}^2)$ since $\ker(f) = (0 :_\omega \mathfrak{a})$ by assumption. The "i.e." part of (3) follows by duality.

Since $(0 :_R \mathfrak{a}) = (0 :_R (\mathfrak{a}\omega))$, $(0 :_R \mathfrak{a}) \subseteq \mathfrak{a}^2$ gives $(0 :_\omega \mathfrak{a}^2) \subseteq \mathfrak{a}\omega$. Hence by (2) and (3), $\ker(h) \subseteq \ker(f)$ proving (4).

In order to prove (2), consider $x_i,\ y_i \in \omega$ $(1 \leq i \leq n)$ such that $\sum_{i=1}^n f(x_i)y_i \in \ker(h) \cap \mathfrak{a}\omega$. We want to show that $\sum_{i=1}^n f(x_i)y_i \in \ker(f)$, i.e., $\sum_{i=1}^n f(x_i)f(y_i) = 0$. Since $\sum_{i=1}^n f(x_i)y_i \in \ker(h)$, we have $0 = h(\sum_{i=1}^n f(x_i)y_i) = \sum_{i=1}^n f(x_i)[f(y_i) + f^*(y_i)]$. Thus, for every $w \in \omega$, $0 = \sum_{i=1}^n f(x_i)[f(y_i) + f^*(y_i)]w = \sum_{i=1}^n f(x_i)[f(y_i)w + f(w)y_i]$ and hence by Remark 2.26 with $M = \omega$, $2\sum_{i=1}^n f(x_i)f(y_i)w = 0$. Since 2 is invertible in R and ω is a faithful R-module, this forces $\sum_{i=1}^n f(x_i)f(y_i) = 0$. $\square$

Remark 2.28. Huneke and Vraciu prove the above proposition for $\mathfrak{a} = \mathfrak{m}$ in [12].

A Ring Structure on the Canonical Module

By the Hom $-\otimes$ adjointness, we see that

$$\operatorname{Hom}_R(\omega \otimes_R \omega, \omega) \simeq \operatorname{Hom}_R(\omega, \operatorname{Hom}_R(\omega, \omega)) \simeq \operatorname{Hom}_R(\omega, R) = \omega^*.$$

Thus, given a map $f \in \omega^*$, we get a map $\tilde{f} : \omega \otimes_R \omega \longrightarrow \omega$ defined by $\tilde{f}(x \otimes y) = f(x)y$. Thus, each $f \in \omega^*$ induces an R-algebra structure (without a unit) on ω. In general, ω need not be a commutative ring under this operation. In fact, it is clear that the multiplication on ω induced by f is commutative if and only if f satisfies Teter's condition.

We can put a ring structure on $S := R \oplus \omega$, with addition defined componentwise and multiplication defined as follows: For (s,x), (t,y) in S,

$$(s,x)(t,y) = (st, sx + ty + f(x)y).$$

Note that S is the algebra obtained by attaching a unit to the R-algebra ω with multiplication induced by f. The ring S is a commutative ring if and only if f satisfies Teter's condition. Moreover, in this case, S is an Artinian local ring with maximal ideal $\mathfrak{m} \oplus \omega$. When f is the zero map, S is the usual idealization of ω into R.

2.4 Algebra Retracts and Gorenstein Colength

Definition 2.29. Let $(R, \mathfrak{m}, \mathsf{k})$ be a commutative Noetherian ring and $\mathfrak{a}$ an ideal in R. We say that a subring T of R is an algebra retract of R with respect to $\mathfrak{a}$ if the map $\pi \circ i : T \longrightarrow R/\mathfrak{a}$ is an isomorphism, where $i : T \longrightarrow R$ is the inclusion and $\pi : R \longrightarrow R/\mathfrak{a}$ is the natural projection.

Remark 2.30. Let R, $\mathfrak{a}$ and T be as in the above definition. We see that the condition $\pi \circ i$ is surjective forces $R = i(T) + \mathfrak{a}$ and the condition $\pi \circ i$ is injective forces $i(T) \cap \mathfrak{a} = 0$. Thus the condition that $\pi \circ i$ is an isomorphism forces $R = i(T) \oplus \mathfrak{a}$. Identifying T with $i(T)$, we see that $R = T \oplus \mathfrak{a}$ as a T-module.

Remark 2.31. Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring such that 2 is invertible in R. Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal in R such that there is a surjective map $f : M \twoheadrightarrow \mathfrak{a}$ with $\mathfrak{a}(\ker f) = 0$.

One can define a multiplicative structure on M as follows: For x, $y \in M$, define $x * y = (f(x)y + f(y)x)/2$. This multiplication is associative by ($\sharp$) in Remark 2.26. Thus M is a ring (without a unit) with multiplication induced by f.

Further, if T is an algebra retract of R with respect to $\mathfrak{a}$, then one can put a ring structure on $S := T \oplus M$, with addition defined componentwise and multiplication defined as follows: For (s,x), (t,y) in S,

$$(s,x)(t,y) = (st, sx + ty + x * y) = \left(st, sx + ty + \frac{f(y)x + f(x)y}{2}\right).$$

Note that S is the algebra obtained by attaching a unit to the T-algebra M with multiplication induced by f. The ring S is a commutative ring. Moreover, S is an Artinian local ring with maximal ideal $\mathfrak{m}_T \oplus M$, where $\mathfrak{m}_T = \mathfrak{m} \cap T$.

The following theorem and Theorem 2.34 are the main theorems proved in [1].

Theorem 2.32. *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring with canonical module ω. Let $\mathfrak{a}$ be a self-dual ideal in R such that $(0 :_R \mathfrak{a}) \subset \mathfrak{a}^2$ and T be an algebra retract of R with respect to $\mathfrak{a}$. Assume further that 2 is invertible in R. Then $g(R) \leq \lambda(R/\mathfrak{a})$.*

Remark 2.33. When $\mathfrak{a} = \mathfrak{m}$, the above hypothesis says that R contains k and that $\mathrm{soc}(R) \subseteq \mathfrak{m}^2$. Huneke and Vraciu prove the theorem in this case in [12].

Proof of Theorem 2.32. Note that since $\mathfrak{a}$ is self-dual, there is a surjective map $f : \omega \longrightarrow\!\!\!\!\rightarrow \mathfrak{a}$ such that $\ker(f) = (0 :_\omega \mathfrak{a})$ by Lemma 2.13. We prove the theorem by constructing a Gorenstein Artin local ring S mapping onto R such that $\lambda(S) - \lambda(R) = \lambda(R/\mathfrak{a})$.

Set $S := T \oplus \omega$. Then S is an Artinian local ring with operations as in Remark 2.31. Define $\phi : S \longrightarrow R$ as $\phi(t,x) = t + f(x)$. Then ϕ is a ring homomorphism and it follows from Remark 2.30 that ϕ is surjective.

We now claim that S is Gorenstein. It is enough to prove that $\lambda(\mathrm{soc}(S)) = 1$. We prove this by showing that $\mathrm{soc}(S) \subseteq (0 :_\omega \mathfrak{m})$ which is a one dimensional vector space over k.

Let $(t,x) \in \mathrm{soc}(S)$ for some $t \in T$ and $x \in \omega$. For each $y \in \omega$, we have $2ty + f(x)y + f(y)x = 0$. Letting y vary over $\ker(f)$, we see that $(2t - x)y = 0$ for all $y \in \ker(f)$, i.e., $(2t - f(x)) \in (0 :_R \ker(f)) = \mathfrak{a}$. Thus $t \in \mathfrak{a}$ which implies that $t = 0$ since by Remark 2.30, $T \cap \mathfrak{a} = 0$.

Now $(0,x)(0,y) = 0$ for all $y \in \omega$ yields $f(x)y + f(y)x = 0$. Thus, if $h : \omega \longrightarrow R$ is defined as in Proposition 2.27, then $h(x)y = f(x)y + f^*(x)y = f(x)y + f(y)x = 0$ for all $y \in \omega$. Since ω is a faithful R-module, this implies that $h(x) = 0$, i.e., $x \in \ker(h)$. Therefore, by Proposition 2.27(4), the hypothesis $(0 :_R \mathfrak{a}) \subset \mathfrak{a}^2$ gives $x \in \ker(f)$.

Let $m \in \mathfrak{m}$. By Remark 2.30, we can write $m = t + a$ for some $t \in \mathfrak{m} \cap T$ and $a \in \mathfrak{a}$. Since $x \in \ker(f) = (0 :_\omega \mathfrak{a})$, $a \cdot x = 0$. Moreover, since $(0,x) \in \mathrm{soc}(S)$, $(0,x)(t,0) = 0$ gives $t \cdot x = 0$. Thus $m \cdot x = 0$ for all $m \in \mathfrak{m}$ proving the theorem. $\square$

2.5 When is $g(R) \leq 2$?

Notation: We use the following notation in the proof of Theorem 2.34:

Let R be any ring and M and N be two R-modules. Let $m_i \in M$ and $n_i \in N$ for $1 \leq i \leq n$. We use the notation $(x_1, \dots, x_n) \overset{\bullet}{\otimes} (y_1, \dots, y_n)$ to denote $\Sigma(x_i \otimes y_i)$.

Theorem 2.34. *With notation as in Setup 2.1, let $\mathfrak{a}$ be an ideal in T, $\mathfrak{b} \subseteq \mathfrak{d} \subseteq \mathfrak{a}$ an ideal generated by a system of parameters such that*

a) *there is an injective map $\overline{\mathfrak{d}} \overset{\phi}{\hookrightarrow} \overline{\mathfrak{a}}^{\vee}$ satisfying $\phi(\overline{x})(\overline{y}) = \phi(\overline{y})(\overline{x})$ for all $x, y \in \mathfrak{d}$,*

b) $\mathfrak{b} \subseteq \mathfrak{a}\mathfrak{d}$ *and*

c) $(\mathfrak{b} :_T \mathfrak{a}) \subseteq \mathfrak{d}$.

Then there is a Gorenstein Artin ring S mapping onto R such that $\lambda(S) - \lambda(R) = \lambda(R/\overline{\mathfrak{a}})$, i.e., $g(R) \leq \lambda(R/\overline{\mathfrak{a}})$.

Proof. The map $\phi \in \mathrm{Hom}_R(\overline{\mathfrak{d}}, \mathrm{Hom}_R(\overline{\mathfrak{a}}, \omega))$ gives a map $\tilde{\phi} \in \mathrm{Hom}_R(\overline{\mathfrak{d}} \otimes_R \overline{\mathfrak{a}}, \omega)$ defined by $\tilde{\phi}(\overline{x} \otimes \overline{y}) = \phi(\overline{x})(\overline{y})$ for any $x \in \mathfrak{d}, y \in \mathfrak{a}$, by the Hom-$\otimes$ adjointness. The hypothesis implies that $\tilde{\phi}(\overline{x} \otimes \overline{y}) = \tilde{\phi}(\overline{y} \otimes \overline{x})$ for $x, y \in \mathfrak{d}$. We have a natural map $\pi : \mathfrak{d}/\mathfrak{b} \otimes \mathfrak{a}/\mathfrak{b} \longrightarrow \mathfrak{d}\mathfrak{a}/\mathfrak{a}\mathfrak{b}$ defined by $\pi((x+\mathfrak{b}) \otimes (y+\mathfrak{b})) = (xy + \mathfrak{a}\mathfrak{b})$.

Consider

$$\begin{tikzcd}
\mathfrak{d}/\mathfrak{b} \otimes \mathfrak{a}/\mathfrak{b} \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi"] & \omega \\
\mathfrak{d}\mathfrak{a}/\mathfrak{a}\mathfrak{b} \arrow[r, "="] & \mathfrak{d}\mathfrak{a}/\mathfrak{a}\mathfrak{b} \arrow[u, dashed, "\widehat{\phi}"']
\end{tikzcd}$$

We claim that there is a map $\widehat{\phi} : \mathfrak{d}\mathfrak{a}/\mathfrak{a}\mathfrak{b} \to \omega$ such that :

(1) the above diagram commutes,

(2) $\mathfrak{b} = (\mathfrak{c} :_T \mathfrak{a})$, where $\mathrm{Ker}(\widehat{\phi}|_{(\mathfrak{b}/\mathfrak{a}\mathfrak{b})}) =: \mathfrak{c}/\mathfrak{a}\mathfrak{b}$,

(3) $S := T/\mathfrak{c}$ is Gorenstein and

(4) $\lambda(S) - \lambda(R) = \lambda(T/\mathfrak{a})$.

In order to prove (1), it is enough to prove that $\mathrm{Ker}(\pi)$ is generated by elements in $\mathfrak{d}/\mathfrak{b} \otimes \mathfrak{a}/\mathfrak{b}$ of the form $(x+\mathfrak{b}) \otimes (y+\mathfrak{b}) - (y+\mathfrak{b}) \otimes (x+\mathfrak{b})$, for $x, y \in \mathfrak{d}$. In such a case $\tilde{\phi}$ restricts to $\widehat{\phi}$.

Let $\mathfrak{d}$ be minimally generated by the regular sequence $x_1, \ldots, x_n$. Let $\Sigma(\bar{k_i} \otimes \bar{a}_i)$ be an element of $\mathrm{Ker}(\pi)$, where $\bar{x}$ denotes $x+\mathfrak{b}$. Since $k_i \in \mathfrak{d}$, without loss of generality we may assume that $\Sigma(\bar{k_i} \otimes \bar{a}_i) = \Sigma_{i=1}^{n}(\bar{x}_i \otimes \bar{a}_i) \in \mathrm{Ker}(\pi)$. Thus $\pi(\Sigma_{i=1}^{n}(\bar{x}_i \otimes \bar{a}_i)) = \Sigma\overline{x_i a_i} = 0$ in $\mathfrak{d}\mathfrak{a}/\mathfrak{a}\mathfrak{b}$. Hence $\Sigma_{i=1}^{n} x_i a_i \in \mathfrak{a}\mathfrak{b}$. Since $\mathfrak{b} \subseteq \mathfrak{d}\mathfrak{a}$, there are elements $u_{ij}, v_j \in \mathfrak{a}$ such that $\Sigma_{i=1}^{n} x_i a_i = \Sigma_{j=1}^{m}(\Sigma_{i=1}^{n} x_i u_{ij}) v_j$ in T, where $\Sigma(x_i u_{ij}) \in \mathfrak{b}$. Hence $\Sigma_{i=1}^{n}(\Sigma_j(u_{ij}v_j) - a_i)x_i = 0$ in T.

Since $x_1, \ldots, x_n$ is a regular sequence in T, we can write

$$(a_1 - \Sigma_j(u_{1j}v_j), \ldots, a_n - \Sigma_j(u_{nj}v_j)) = \Sigma_{i<j} t_{ij}(x_j e_i - x_i e_j) \qquad (i)$$

for some $t_{ij} \in T$, where $\{e_i\}_{i=1}^{n}$ is the standard basis of T^n. Then we have

$$(\bar{x}_1, \ldots, \bar{x}_n) \overset{\bullet}{\otimes} (\Sigma_j(\overline{u_{1j}v_j}), \ldots, \Sigma_j(\overline{u_{nj}v_j}))$$

$$= (\Sigma_i(\overline{u_{i1}x_i}), \ldots, \Sigma_i(\overline{u_{im}x_i})) \overset{\bullet}{\otimes} (\bar{v}_1, \ldots, \bar{v}_m) = 0 \qquad (ii)$$

since $\Sigma_i(u_{ij}x_i) \in \mathfrak{b}$ for each j. Thus, using Equations (i) and (ii), we see that

$$\Sigma_{i=1}^{n}(\bar{x}_i \otimes \bar{a}_i) = (\bar{x}_1, \ldots, \bar{x}_n) \overset{\bullet}{\otimes} (\bar{a}_1, \ldots, \bar{a}_n) =$$

$$(\bar{x}_1, \ldots, \bar{x}_n) \overset{\bullet}{\otimes} \Sigma_{i<j} \bar{t_{ij}}(0, \ldots, \bar{x}_j, \ldots, -\bar{x}_i, \ldots, 0) = \Sigma \bar{t_{ij}}(\bar{x}_i \otimes \bar{x}_j - \bar{x}_j \otimes \bar{x}_i)$$

verifying (1).

We now have a map $\mathfrak{d}\mathfrak{a}/\mathfrak{a}\mathfrak{b} \xrightarrow{\widehat{\phi}} \omega$ where $\widehat{\phi}(\overline{\Sigma a_i b_i}) = \Sigma\phi(\overline{a_i})(\overline{b_i})$. Restrict $\widehat{\phi}$ to $\mathfrak{b}/\mathfrak{a}\mathfrak{b}$, call it ψ. Let $\mathfrak{c} \subseteq T$ be defined by $\mathfrak{c}/\mathfrak{a}\mathfrak{b} = \mathrm{Ker}(\psi)$. Then $(\mathfrak{c} :_T \mathfrak{a}) = \mathfrak{b}$ which can be seen as follows: Let $u \in (\mathfrak{c} :_T \mathfrak{a})$. Note that the hypothesis $(\mathfrak{b} :_T \mathfrak{a}) \subseteq \mathfrak{d}$ gives $u \in \mathfrak{d}$. For any $a \in \mathfrak{a}$, we have $0 = \psi(\overline{ua}) = \phi(\bar{u})(\bar{a})$. Hence $\phi(\bar{u}) = 0$ in $\overline{\mathfrak{a}}^\vee$. Since ϕ is an injective map, $u \in \mathfrak{b}$ as claimed in (2).

The map ψ induces an inclusion $\mathfrak{b}/\mathfrak{c} \hookrightarrow \omega$. Since $\mathfrak{b} = (\mathfrak{c} :_T \mathfrak{a})$, $(\mathfrak{c} :_T \mathfrak{m}_T) \subseteq \mathfrak{b}$, i.e., $(\mathfrak{c} :_T \mathfrak{m}_T)/\mathfrak{c} \simeq \operatorname{soc}(T/\mathfrak{c}) \subseteq \mathfrak{b}/\mathfrak{c}$. Therefore the inclusion $(\mathfrak{c} :_T \mathfrak{m}_T)/\mathfrak{c} \hookrightarrow \operatorname{soc}(\omega)$, yields $\lambda(\operatorname{soc}(T/\mathfrak{c})) = 1$ since ω has a one-dimensional socle. Thus $S := T/\mathfrak{c}$ is Gorenstein proving (3).

Lastly, since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{c}$ and $\mathfrak{b} \subseteq \mathfrak{a}$, $\mathfrak{b}^2 \subseteq \mathfrak{c}$. Therefore $(\mathfrak{b}/\mathfrak{c})^2 = 0$ in S and $R \simeq S/(\mathfrak{b}/\mathfrak{c})$. Now, by Proposition 2.15(1), $\omega \simeq (0 :_S (\mathfrak{b}/\mathfrak{c})) \simeq \mathfrak{a}/\mathfrak{c}$ since $T/\mathfrak{c}$ is a Gorenstein ring. Thus the image of ω in R under the natural map from S to R is $\mathfrak{a}/\mathfrak{b}$, and $(\mathfrak{b}/\mathfrak{c})^2 = 0$ in S. Hence by Lemma 2.17, $\lambda(S) - \lambda(R) = \lambda(R/\overline{\mathfrak{a}}) = \lambda(T/\mathfrak{a})$ which proves (4) and completes the proof of the theorem. □

With notation as in Theorem 2.34, if $\mathfrak{a} = \mathfrak{d}$, then condition (c) in the above theorem follows from condition (b). Thus we have the following:

Corollary 2.35. *With notation as in Setup 2.1, let $\mathfrak{a}$ be an ideal in T generated by a system of parameters such that $\mathfrak{b} \subseteq \mathfrak{a}^2$. Let $\phi : \overline{\mathfrak{a}} \longrightarrow \overline{\mathfrak{a}}^\vee$ be an isomorphism satisfying $\phi(\bar{x})(\bar{y}) = \phi(\bar{y})(\bar{x})$ for all $x, y \in \mathfrak{a}$. Then $g(R) \leq \lambda(R/\overline{\mathfrak{a}})$.*

We recover Teter's theorem from Corollary 2.35 by taking $\mathfrak{a} = \mathfrak{m}$. With some additional hypothesis, we see in the next corollary that we can get rid of Teter's condition on the map ϕ in Theorem 2.34, just as Huneke and Vraciu did in the case of Teter's theorem.

Corollary 2.36. *With notation as in Setup 2.1, let $\mathfrak{a}$ be an ideal in T, $\mathfrak{d} \subseteq \mathfrak{a}$ an ideal generated by a system of parameters such that*

a) $\overline{\mathfrak{a}} \simeq \overline{\mathfrak{a}}^\vee$.

b) $\mathfrak{b} \subseteq \mathfrak{a}\mathfrak{d}$ *and*

c) $(\mathfrak{b} :_T \mathfrak{a}) \subseteq \mathfrak{d} \cap \mathfrak{a}^2$.

Further assume that 2 is invertible in R. Then $g(R) \leq \lambda(R/\overline{\mathfrak{a}})$.

Proof. Since $\overline{\mathfrak{a}} \simeq \overline{\mathfrak{a}}^\vee$, by Lemma 2.13, $\overline{\mathfrak{a}} = f(\omega)$ for some $f \in \omega^*$ satisying the condition $\ker(f) \cdot f(\omega) = 0$. By (3), $(\mathfrak{b} :_T \mathfrak{a}) \subseteq \mathfrak{a}^2$, i.e., $(0 :_R \overline{\mathfrak{a}}) \subseteq \overline{\mathfrak{a}}^2$. Hence by Proposition 2.27, there is a map $h \in \omega^*$ satisfying Teter's condition such that $h(\omega) = \overline{\mathfrak{a}}$. By Lemma 2.13, since h satisfies Teter's condition, so does the induced isomorphism $\phi : \overline{\mathfrak{a}} \xrightarrow{\sim} \overline{\mathfrak{a}}^\vee$.

Thus ϕ restricted to $\overline{\mathfrak{d}}$ satisfies condition (1) of Theorem 2.34. Hence the conclusion of the corollary follows from Theorem 2.34. □

By taking $\mathfrak{a} = \mathfrak{d}$ in the above corollary, we get the following

Corollary 2.37. *With notation as in Setup 2.1, let* $\mathfrak{a}$ *be an ideal in* T *generated by a system of parameters. Furthermore, assume that* $\overline{\mathfrak{a}} \simeq \overline{\mathfrak{a}}^\vee$*, 2 is invertible in* R *and* $\mathfrak{b} \subseteq \mathfrak{a}^3$*. Then* $g(R) \leq \lambda(R/\overline{\mathfrak{a}})$.

The following is really a corollary, but is important enough to be accorded the status of a theorem.

Theorem 2.38. *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be an Artinian local ring. Write* $R \simeq T/\mathfrak{b}$ *where* $(T, \mathfrak{m}_T, \mathsf{k})$ *is a regular local ring and* $\mathfrak{b}$ *is an* $\mathfrak{m}_T$*-primary ideal. Let* $\bar{}$ *denote going modulo* $\mathfrak{b}$*. Suppose that* $\mathfrak{b} \subseteq \mathfrak{m}_T^6$ *and 2 is invertible in R. Then the following are equivalent:*

i) $g(R) \leq 2$.

ii) *There exists a self-dual ideal* $\overline{\mathfrak{a}} \subseteq R$ *such that* $\lambda(R/\overline{\mathfrak{a}}) \leq 2$.

Proof. (i) $\Rightarrow$ (ii) follows from the fundamental inequality.

(ii) $\Rightarrow$ (i): If $\lambda(T/\mathfrak{a}) \leq 1$, then by the Huneke-Vraciu Theorem (Theorem 0.3), $g(R) \leq 1$, since $\mathfrak{b} \subseteq \mathfrak{m}_T^6$ implies that $\mathrm{soc}(R) \subseteq \mathfrak{m}^2$. If $\lambda(T/\mathfrak{a}) = 2$, then $\mathfrak{a}$ is generated by a system of parameters and $\mathfrak{b} \subseteq \mathfrak{m}_T^6$ forces $\mathfrak{b} \subseteq \mathfrak{a}^3$. Hence, by Corollary 2.37, $g(R) \leq 2$. $\square$

Remark 2.39. Let the hypothesis be as in Theorem 2.38. By combining the conclusions of the Huneke-Vraciu theorem and Theorem 2.38, we see that $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\} = g(R)$ when either of the two quantities is at most two. Further, it follows from Theorem 2.38 and Corollary 2.18 that if $g(R) = 3$, then so is $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\}$. Thus we see that in this case, Question 2.19 has a positive answer if either $g(R) \leq 3$ or $\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\} \leq 2$.

2.6 A Detour into Golod Homomorphisms

Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring and M be a finitely generated module with a minimal free resolution

$$\cdots \to R^{b_d} \to \cdots \to R^{b_1} \to R^{b_0} \to M \to 0.$$

Since the resolution is minimal, we have the ith Betti number of M over R, $b_i^R(M)$ (or simply b_i) = $\dim_{\mathsf{k}}(\mathrm{Tor}_i^R(\mathsf{k}, M))$. We define the Poincare series of M over R, denoted $P_R^M(t)$ as

$$P_R^M(t) = \sum_{i \geq 0} b_i t^i.$$

Definition 2.40. *Let $\phi : (S, \mathfrak{m}_S, \mathsf{k}) \longrightarrow (R, \mathfrak{m}, \mathsf{k})$ be a local homomorphism of Noetherian local rings. We say that ϕ is Golod if the following equality holds:*

$$P_R^{\mathsf{k}}(t) = \frac{P_S^{\mathsf{k}}(t)}{1 - t(P_S^R(t) - 1)}.$$

Note that since R is a quotient of S and k is a quotient of S and R, $b_0^S(R) = 1$, $b_0^S(\mathsf{k}) = 1$ and $b_0^R(\mathsf{k}) = 1$. Thus, if $\phi : S \longrightarrow R$ is Golod, the following equalities are forced:

$$b_1^S(\mathsf{k}) = b_1^R(\mathsf{k}), \quad b_2^S(\mathsf{k}) = -b_1^S(R) + b_2^R(\mathsf{k}), \quad b_3^S(\mathsf{k}) = -b_2^S(R) - b_1^S(R)b_1^R(\mathsf{k}) + b_3^R(\mathsf{k});$$

$$b_4^S(\mathsf{k}) = -b_3^S(R) - b_2^S(R)b_1^R(\mathsf{k}) - b_1^S(R)b_2^R(\mathsf{k}) + b_4^R(\mathsf{k}), \text{ etc.}$$

In their paper [3], Avramov and Levin show that if S is a Gorenstein Artin local ring and $R \simeq S/\operatorname{soc}(S)$, then the natural projection $S \longrightarrow\!\!\!\!\rightarrow R$ is a Golod homomorphism. In particular, their result yields the following: if $g(R) = 1$ and S is a Gorenstein Artin local ring mapping onto R via a ring homomorphism ϕ such that $\lambda(S) - \lambda(R) = 1$, then the map $S \xrightarrow{\phi}\!\!\!\!\rightarrow R$ is a Golod homomorphism. This raises the following question:

Question 2.41. *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring. Let S be a Gorenstein Artin local ring mapping onto R via a ring homomorphism ϕ such that $\lambda(S) - \lambda(R) = g(R)$. Is $S \xrightarrow{\phi}\!\!\!\!\rightarrow R$ a Golod homomorphism?*

This need not be true in general. The following example shows that Question 2.41 does not have a positive answer even in the case where $g(R) = 2$.

Example 2.42. Let $R = \mathbb{Q}[x,y,z]/\mathfrak{b}$ and $S = \mathbb{Q}[x,y,z]/\mathfrak{c}$ where $\mathfrak{c} = (y^2 - xz, xyz, x^2z, x^2y - z^3, x^3 - yz^2)$ and $\mathfrak{b} = \mathfrak{c} + (yz^2)$. Clearly $S \longrightarrow\!\!\!\!\rightarrow R$ since $\mathfrak{c} \subseteq \mathfrak{b}$. Moreover, $\lambda(S) - \lambda(R) = 2$ which proves that $g(R) \leq 2$.

Let ω be the canonical module of R. We can compute $\omega^*(\omega)$ using Macaulay 2 as in Remark 2.23. Thus, we have $\omega^*(\omega) = (x,z)$, which yields $\lambda(R/\omega^*(\omega)) = 2$. By the fundamental inequalities, this shows that $g(R) \geq 2$. Thus $g(R) = 2$.

It is easy to see that the map $S \longrightarrow R$ is not Golod. In this case

$$P^R_S(t) = 1 + t + 2t^2 + 5t^3 + 13t^4 + \cdots; \quad P^{\mathsf{k}}_S(t) = 1 + 3t + 8t^2 + 21t^3 + 55t^4 + \cdots \text{ and}$$

$$P^{\mathsf{k}}_R(t) = 1 + 3t + 9t^2 + 25t^3 + 71t^4 + \cdots.$$

Since $-b^S_3(R) - b^S_2(R)b^R_1(\mathsf{k}) - b^S_1(R)b^R_2(\mathsf{k}) + b^R_4(\mathsf{k}) = -5 - 2*3 - 1*9 + 71 = 51 < 55 = b^S_4(\mathsf{k})$, the map $S \longrightarrow R$ is not Golod.

Chapter 3

Computing Gorenstein Colength

Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring. The socle of R, $\operatorname{soc}(R)$, is a direct sum of finitely many copies of k, hence it is isomorphic to $\operatorname{soc}(R)^\vee$, i.e., $\operatorname{soc}(R)$ is a self-dual ideal. Hence a particular case of Question 2.20 is the following:

Question 3.1. Is there a Gorenstein Artin local ring S mapping onto R such that $\lambda(S) - \lambda(R) = \lambda(R/\operatorname{soc}(R))$?

A weaker question one can ask is the following:

Question 3.2. Is $g(R) \leq \lambda(R/\operatorname{soc}(R))$?

We answer Question 3.2 in two cases in this chapter. In section 1, we show that if T is a power series ring over a field and $\mathfrak{d} = (f_1, \ldots, f_d)$ is an ideal generated by a system of parameters, then $g(T/\mathfrak{d}^n) \geq \lambda(T/\mathfrak{d}^{n-1})$. Further, if the residue field of T has characteristic zero, we construct a Gorenstein Artin local ring S mapping onto $T/\mathfrak{d}^n$ such that $\lambda(S) - \lambda(T/\mathfrak{d}^n) = \lambda(T/\mathfrak{d}^{n-1})$ using a theorem of L. Ried, L. Roberts and M. Roitman proved in [19]. This shows that $g(T/\mathfrak{d}^n) = \lambda(T/\mathfrak{d}^{n-1})$. In particular, this proves that $R = T/\mathfrak{d}^n$ satisfies the inequality in Question 3.2.

In [16], Kleppe, Migliore, Miró-Roig, Nagel and Peterson show that $\mathfrak{d}^n$ can be linked to $\mathfrak{d}^{n-1}$ via Gorenstein ideals in 2 steps and hence to $\mathfrak{d}$ in $2(n-1)$ steps. In section 2, we use the ideal corresponding to the Gorenstein ring constructed in section 2, to show that $\mathfrak{d}^n$ can be directly liked to $\mathfrak{d}^{n-1}$ and hence to $\mathfrak{d}$ in $(n-1)$ steps.

When R is an Artinian quotient of a two-dimensional regular local ring with an infinte residue field, we use a formula due to Hoskin and Deligne (Theorem 3.37) in order to answer Question 3.2 in section 3. The results in this chapter are the main ingredients in [2].

3.1 Powers of Ideals Generated by a System of Parameters

One of the main theorems we prove in this section is the following:

Theorem 3.3. *Let $T = \mathsf{k}[|X_1,\ldots,X_d|]$ be a power series ring over a field k of characteristic zero. Let $f_1,\ldots,f_d$ be a system of parameters and $R = T/(f_1,\ldots,f_d)^n$. Then $g(R) = \lambda(T/(f_1,\ldots,f_d)^{n-1})$.*

In order to prove this, we first prove Theorems 3.4 and 3.13, which proves Theorem 3.3 in the special case where $(f_1,\ldots,f_d) = (x_1,\ldots,x_d)$. We then use the fact that T is flat over $T' = \mathsf{k}[|f_1,\ldots,f_d|]$ to prove Theorem 3.3 in general.

Theorem 3.4. *Let $T = \mathsf{k}[|X_1,\ldots,X_d|]$ be a power series ring over a field k with unique maximal ideal $\mathfrak{m}_T = (X_1,\ldots,X_d)$. Let $R = T/\mathfrak{m}_T^n$. . Then $\omega^*(\omega) = \operatorname{soc}(R) = \mathfrak{m}_T^{n-1}/\mathfrak{m}_T^n$.*

Proof. In order to prove this, we show that if $\phi \in \operatorname{Hom}(\omega,R)$, then $\phi(\omega) \subseteq \operatorname{soc}(R)$. Since we already know that $\operatorname{soc}(R) \subseteq \omega^*(\omega)$, this will prove the theorem.

Note that we can consider R to be the quotient of the polynomial ring $\mathsf{k}[X_1,\ldots,X_d]$ by $(X_1,\ldots,X_d)^n$. Thus change notation so that $T = \mathsf{k}[X_1,\ldots,X_d]$ and $\mathfrak{m}_T = (X_1,\ldots,X_d)$ is its unique homogenous maximal ideal.

The injective hull of k over T is $\mathsf{k}[X_1^{-1},\ldots,X_d^{-1}]$, where the multiplication is defined by

$$(X_1^{a_1}\cdots X_d^{a_d})\cdot(X_1^{-b_1}\cdots X_d^{-b_d})=\begin{cases} X_1^{a_1-b_1}\cdots X_d^{a_d-b_d} & \text{if } a_i\le b_i \text{ for all } i\\ 0 & \text{otherwise}\end{cases}$$

and extended linearly (e.g., see [17]).

Let $\mathfrak{b}=\mathfrak{m}_T^n$. We know that the canonical module ω of R is isomorphic to the injective hull of the residure field of R. Thus $\omega\simeq E_R(\mathsf{k})\simeq \operatorname{Hom}_R(R,E_T(\mathsf{k}))\simeq(0:_{\mathsf{k}[X_1^{-1},\ldots,X_d^{-1}]}\mathfrak{b})$. Note that $\mathfrak{b}\cdot(X_1^{-a_1}\cdots X_d^{-a_d})=0$ whenever $a_i\ge 0$ and $n>\sum a_i$. Since $\lambda(\omega)=\lambda(R)$, we conclude that

$$\omega\simeq \mathsf{k}\text{-span of }\{X_1^{-a_1}\cdots X_d^{-a_d}: a_i\ge 0; n>\sum a_i\}.$$

Observe that ω is generated by $\{X_1^{-a_1}\cdots X_d^{-a_d}\colon \sum a_i=n-1\}$ as an R-module. Let $\phi\in\omega^*$. We will now show that $\phi(X_1^{-a_1}\cdots X_d^{-a_d})\in\operatorname{soc}(R)$ by induction on a_1. Let $w=X_1^{-a_1}\cdots X_d^{-a_d}$, $\sum a_i=n-1$.

If $a_1=0$, then $X_1\cdot w=0$. Hence $\phi(w)\in(0:_R X_1)=\operatorname{soc}(R)$. If not, then $X_1w=X_2(X_1^{-(a_1-1)}X_2^{-(a_2+1)}\cdots X_d^{-a_d})$. We have $\phi(X_1^{-(a_1-1)}X_2^{-(a_2+1)}\cdots X_d^{-a_d})\in\operatorname{soc}(R)$ by induction. Thus $X_2\phi(X_1^{-(a_1-1)}X_2^{-(a_2+1)}\cdots X_d^{-a_d})=0$ which yields $X_1\phi(w)=0$. But $(0:_R X_1)=\operatorname{soc}(R)$, which proves that $\phi(\omega)\subseteq\operatorname{soc}(R)$. $\square$

Since we know that $\lambda(R/(\omega^*(\omega))\le g(R)$ by the fundamental inequalities, we immediately get the following:

Corollary 3.5. *With notation as in Thoerem 3.4,* $g(R)\ge\lambda(R/\operatorname{soc}(R))$.

Remark 3.6. Let $u_0=X_1^{-a_1}\cdots X_d^{-a_d}$, $\sum a_i=n-1$, be a minimal generator of ω. If $a_1=0$, then $X_1u_0=0$. If $a_1>0$, then $u_1=X_1^{-(a_1-1)}X_2^{-(a_2+1)}\cdots X_d^{-a_d}$ is also a minimal generator of ω. Note that $X_1u_0 = X_2u_1$. Continuing in a similar fashion, we get a sequence of minimal generators of ω, $u_i = X_1^{-(a_1-i)}X_2^{-(a_2+i)}\cdots X_d^{-a_d}$, $i = 0,\ldots,a_1$ such that $X_1u_i = X_2u_{i+1}$ for $i < a_1$ and $X_1u_{a_1} = 0$.

We prove the reverse inequality in Theorem 3.13 by constructing a Gorenstein Artin ring S ring mapping onto R such that $\lambda(S) - \lambda(R) = \lambda(R/\mathrm{soc}(R))$. We use the following notation in this section.

Let us prove some basic facts before we prove Theorems 3.13.

Lemma 3.7. *Let $(S,\mathfrak{m},\mathsf{k})$ be a Gorenstein Artin local ring, I a proper ideal in S. Then the following are equivalent:*

i) *$I = (0 :_S f)$, where f is a non-zero element in S.*

ii) *S/I is a Gorenstein Artin local ring.*

Proof. (i) $\Rightarrow$ (ii): Consider the short exact sequence $0 \to S/(0 :_S f) \xrightarrow{\cdot f} S \to S/(f) \to 0$. Since $\mathrm{soc}(S/(0 :_S f))$ is non-zero and maps into $\mathrm{soc}(S)$, which has length 1, $\lambda(\mathrm{soc}(S/(0 :_S f))) = 1$. Thus $S/(0 :_S f)$ is Gorenstein.

(ii) $\Rightarrow$ (i): Since S is Gorenstein, if ω is the canonical module of S/I, then $\omega \simeq 0 :_S I$. If S/I is Gorenstein, then ω is cyclic. Therefore $(0 :_S I) = (f)$ for some non-zero f in S. Since $0 :_S (0 :_S I) = I$, it follows that $I = (0 :_S f)$. $\square$

Proposition 3.8. *Let $S = \mathsf{k}[X_1,\ldots,X_d]/(X_1^{n_1},\ldots,X_d^{n_d})$ be a quotient of the polynomial ring over a field k and f be a non-zero homogeneous element in S of degree s. Then $\mathrm{Max}(S/(0 :_S f)) = \mathrm{Max}(S) - s$.*

Proof. Set $J = (0 :_S f)$. Suppose g is a homogeneous element in S of degree more than $\mathrm{Max}(S) - s$. Then $\deg(fg) > \mathrm{Max}(S)$. Hence if $\deg(g) > \mathrm{Max}(S) - s$, $g \in J$. Thus $\mathrm{Max}(S/J) \leq \mathrm{Max}(S) - s$.

On the other hand, since S is Gorenstein, and its socle is in degree $\mathrm{Max}(S)$, given f of degree s, there is a homogeneous element g such that $(f \cdot g) = \mathrm{soc}(S)$. Thus $g \notin J$. Since $\deg(g) = \mathrm{Max}(S) - s$, $\mathrm{Max}(S/J) \geq \mathrm{Max}(S) - s$. $\square$

The following theorem of Ried, Roberts and Roitman is used in the construction of a Gorenstein Artin local ring S mapping onto R, where R is as in Theorem 3.13. Since the proof is short, we include it for the sake of completeness.

Theorem 3.9 (Reid-Roberts-Roitman([19])). *Let* k *be a field of characteristic zero,* $A = \mathsf{k}[X_1, \ldots, X_d]/(X_1^{n_1}, \ldots, X_d^{n_d}) = \mathsf{k}[x_1, \ldots, x_d]$. *Let* $m \geq 1$ *and* f *be a nonzero homogeneous element in* A *such that* $(x_1 + \cdots + x_d)^m f = 0$. *Then* $\deg(f) \geq (t - m + 1)/2$, *where* $t = \sum_{i=1}^d (n_i - 1)$.

Proof. Let $l = X_1 + \cdots + X_d$ and $F \in \mathsf{k}[X_1, \ldots, X_d]$ be a homogeneous element such that its image in A is f. Note that A is a Gorenstein Artin local ring and $\mathrm{Max}(A) = t$.

Induce on $\deg(f)$. If $\deg(f) = 0$, then $l^m \in (X_1^{n_1}, \ldots, X_d^{n_d})$. Hence every monomial that appears in l^m is in $(X_1^{n_1}, \ldots, X_d^{n_d})$ since it is a monomial ideal. But each monomial of degree m appears in l^m and hence $\mathfrak{m}^m \subseteq (X_1^{n_1}, \ldots, X_d^{n_d})$. This yields $m \geq t + 1$ proving the inequality.

Now suppose that $\deg(f) \geq 1$. We are given that $l^m F \in (X_1^{n_1}, \ldots, X_d^{n_d})$. Taking the partial derivative with respect to X_d, we see that $ml^{m-1}F + l^m \frac{\partial F}{\partial X_d} \in (X_1^{n_1}, \ldots, X_d^{n_d - 1})$. Multiplying by l gives $l^{m+1} \frac{\partial F}{\partial X_d} \in (X_1^{n_1}, \ldots, X_d^{n_d - 1})$. By induction, this forces

$$\deg\left(\frac{\partial F}{\partial X_d}\right) \geq \left(\sum_{i=1}^{d-1} (n_i - 1) + n_d - 2 - (m+1) + 1\right)/2,$$

i.e., $\deg(F) - 1 \geq ((t-1) - (m+1) + 1)/2$. This gives us the required inequality. $\square$

Remark 3.10. Let $T = \mathsf{k}[X_1, \ldots, X_d]$ be a polynomial ring over k and $\mathfrak{m}_T = (X_1, \ldots, X_d)$ be its unique homogeneous maximal ideal. Let f be a homogeneous element and $\mathfrak{c} =$

$(X_1^n,\dots,X_d^n) :_T f$. Then one can see that $\mathfrak{c} \subseteq \mathfrak{m}_T^n$ if and only if multiplication by f is injective on the i^{th} graded pieces of $T/(X_1^n,\dots,X_d^n)$ for $i < n$.

Proposition 3.11. *Let $T = \mathsf{k}[|X_1,\dots,X_d|]$ be a power series ring over a field k with unique maximal ideal $\mathfrak{m}_T = (X_1,\dots,X_d)$. Let f be a homogeneous element and $\mathfrak{c} = (X_1^n,\dots,X_d^n) :_T f$ be such that $\mathfrak{c} \subseteq \mathfrak{m}^n$. Then the following are equivalent:*

i) $\lambda(\mathfrak{m}_T^n/\mathfrak{c}) = \lambda(T/\mathfrak{m}_T^{n-1})$.

ii) $\mathrm{Max}(T/\mathfrak{c}) = 2(n-1)$.

iii) $\deg(f) = (d-2)(n-1)$.

Proof. Since $\mathrm{Max}(T/(X_1^n,\dots,X_d^n)) = d(n-1)$, (ii) $\Leftrightarrow$ (iii) follows from Proposition 3.8.

Let $R = T/\mathfrak{m}_T^n$ and $S = T/\mathfrak{c}$. Since $T/(X_1^n,\dots,X_d^n)$ is a Gorenstein Artin local ring, so is S by Lemma 3.7. Note that $\mathrm{soc}(R) = \mathfrak{m}_T^{n-1}/\mathfrak{m}_T^n$ and $\lambda(S) - \lambda(R) = \lambda(\mathfrak{m}_T^n/\mathfrak{c})$.

The rings R and S are quotients of the polynomial ring $\mathsf{k}[X_1,\dots,X_d]$ by homogeneous ideals. Thus, both R and S are graded under the standard grading. Since $\mathfrak{c} \subseteq \mathfrak{m}_T^n$,

$$h_S(i) = h_R(i) \text{ for } i < n. \qquad (*)$$

Since S is Gorenstein,

$$h_S(i) = h_S(\mathrm{Max}(S) - i). \qquad (**)$$

Using $(*)$ and $(**)$, we see that the h-vectors of R and S are:

degree i	0	1	2	3	…	n-1
$h_R(i)$	1	d	$\binom{d+1}{2}$	$\binom{d+2}{3}$	…	$\binom{d+n-2}{n-1}$
$h_S(i)$	1	d	$\binom{d+1}{2}$	$\binom{d+2}{3}$	…	$\binom{d+n-2}{n-1}$

degree i	...	Max(S) - (n-1)	Max(S) - (n-2)	...	Max(S) - 1	Max(S)
$h_R(i)$	...	0	0	...	0	0
$h_S(i)$	...	$\binom{d+n-2}{n-1}$	$\binom{d+n-3}{n-2}$	...	d	1

Thus, using $(*)$ and $(**)$, we have

$$\begin{aligned}
\lambda(T/\mathfrak{m}_T^{n-1}) &= h_R(n-2)+h_R(n-3)+\ldots+h_R(0)\\
&= h_S(n-2)+h_S(n-3)+\ldots+h_S(0)\\
&= h_S(\mathrm{Max}(S)-(n-2))+h_S(\mathrm{Max}(S)-(n-3))+\ldots+h_S(\mathrm{Max}(S))\\
&= \textstyle\sum_{i\geq \mathrm{Max}(S)-(n-2)} h_S(i)\\
&\leq \lambda(S)-\lambda(R)=\lambda(\mathfrak{m}_T^n/\mathfrak{c}).
\end{aligned}$$

Moreover, from the above table, equality holds if and only if $\mathrm{Max}(S)-(n-1)=n-1$, proving (i) $\Leftrightarrow$ (ii). $\square$

In the following corollary, we show that $f=(X_1+\cdots+X_d)^{(d-2)(n-1)}$ satisfies the hypothesis of Proposition 3.11.

Corollary 3.12. *Let $T=\mathsf{k}[X_1,\ldots,X_d]$ be a polynomial ring over k, a field of characteristic zero, and $\mathfrak{m}_T=(X_1,\ldots,X_d)$ be its unique homogeneous maximal ideal. Let $\mathfrak{c}_n=(X_1^n,\ldots,X_d^n):l^{(d-2)(n-1)}$, where $l=X_1+\cdots+X_d$. Then $\mathfrak{c}_n\subseteq\mathfrak{m}_T^n$.*

Moreover, $\lambda(\mathfrak{m}_T^n/\mathfrak{c}_n)=\lambda(T/\mathfrak{m}_T^{n-1})$.

Proof. If F is a homogeneous element in T such that $l^mF\in(X_1^n,\ldots,X_d^n)$, then $\deg(F)\geq (d(n-1)-m+1)/2$ by Theorem 3.9. Therefore, for $m=(d-2)(n-1)$, we see that $\deg(F)\geq n-1/2$, i.e. $F\in\mathfrak{m}_T^n$. Thus $(X_1^n,\ldots,X_d^n):(X_1+\cdots X_d)^{(d-2)(n-1)}\subseteq\mathfrak{m}_T^n$.

Moreover, by Proposition 3.11, since $\deg(l^{(d-2)(n-1)})=(d-2)(n-1)$, $\lambda(\mathfrak{m}_T^n/\mathfrak{c}_n)=\lambda(T/\mathfrak{m}_T^{n-1})$. $\square$

Theorem 3.13. *Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field k of characteristic zero, with unique maximal ideal $\mathfrak{m}_T = (X_1, \dots, X_d)$. Let $R := T/\mathfrak{m}_T^n$. Then $g(R) \leq \lambda(R/\operatorname{soc}(R)) = \lambda(T/\mathfrak{m}^{n-1})$.*

Proof. Let $\mathfrak{c}_n = (X_1^n, \dots, X_d^n) :_T l^{(d-2)(n-1)}$, where $l = X_1 + \cdots + X_d$. Let $S = T/\mathfrak{c}_n$. Then S is a Gorenstein Artin local ring mapping onto R. Since $R \simeq \mathsf{k}[X_1, \dots, X_d]/(X_1, \dots, X_d)^n$ and $S \simeq \mathsf{k}[X_1, \dots, X_d]/((X_1^n, \dots, X_d^n) :_T l^{(d-2)(n-1)})$, by Corollary 3.12, $\lambda(S) - \lambda(R) = \lambda(R/\operatorname{soc}(R)) = \lambda(T/\mathfrak{m}_T^{n-1})$. This shows that $g(R) \leq \lambda(R/\operatorname{soc}(R))$. $\square$

Remark 3.14. Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field k of characteristic zero, with unique maximal ideal $\mathfrak{m}_T = (X_1, \dots, X_d)$. From the above proof, we see that the ideal $\mathfrak{c}_n = (X_1^n, \dots, X_d^n) :_T (X_1 + \cdots + X_d)^{(d-2)(n-1)} \subseteq \mathfrak{m}_T^n$ and $\lambda(\mathfrak{m}_T^n/\mathfrak{c}_n) = \lambda(T/\mathfrak{m}_T^{n-1})$. Thus we see that the conclusions of Corollary 3.12 hold for the power series ring $\mathsf{k}[|X_1, \dots, X_d|]$ as well.

Comments:

The ring S constructed in the proof of the theorem does not work in positive characteristic, eg., when $\operatorname{char}(\mathsf{k}) = 2$, for $d = 3, n = 3$, we have $h_R(i) = (1, 3, 6)$ and $h_S(i) = (1, 2, 5, 2, 1)$.

In the following remark, we record some key observations which we will use to prove Theorem 3.3.

Remark 3.15. Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field k. Let $f_1, \dots, f_d$ be a system of parameters. Then $T' = \mathsf{k}[|f_1, \dots, f_d|]$ is a power series ring and T is free over T' of rank $e = \lambda(T/(f_1, \dots, f_d))$. Thus, if $\mathfrak{b}$ and $\mathfrak{c}$ are ideals in T', then $(\mathfrak{c} :_{T'} \mathfrak{b})T = (\mathfrak{c}T :_T \mathfrak{b}T)$ and $\lambda(T/\mathfrak{b}T) = e \cdot \lambda(T'/\mathfrak{b})$.

Firstly, we construct a Gorenstein Artin ring S mapping onto R such that $\lambda(S) - \lambda(R) = \lambda(T/(f_1, \dots, f_d)^{n-1})$ which proves $g(R) \leq \lambda(T/(f_1, \dots, f_d)^{n-1})$. We do this as follows:

Suppose that $\mathrm{char}(\mathsf{k}) = 0$. Let $\mathfrak{d} = (f_1, \ldots, f_d)$, $\mathfrak{c} = (f_1^n, \ldots, f_d^n) :_{T'} l^{(d-2)(n-1)}$, where $l = (f_1 + \cdots + f_d)$. We see that since $(f_1^n, \ldots, f_d^n) :_{T'} l^{(d-2)(n-1)} \subseteq \mathfrak{d}^n$ in T' by Corollary 3.12, the same holds in T by using Remark 3.15. Moreover, $\lambda(\mathfrak{d}^n T/\mathfrak{c}T) = e\lambda(\mathfrak{d}^n/\mathfrak{c})$ and $\lambda(T/\mathfrak{d}^{n-1}T) = e\lambda(T'/\mathfrak{d}^{n-1})$, the length condition in Corollary 3.12 gives $\lambda(\mathfrak{d}^n T/\mathfrak{c}T) = \lambda(T/\mathfrak{d}^{n-1}T)$.

This implies that if $R = T/\mathfrak{d}^n T$, then $S = T/\mathfrak{c}T$ is a Gorenstein Artin ring mapping onto R and $\lambda(S) - \lambda(R) = \lambda(\mathfrak{d}^n T/\mathfrak{c}T) = \lambda(T/\mathfrak{d}^{n-1}T)$. Therefore $g(R) \leq \lambda(T/\mathfrak{d}^{n-1}T)$. Thus as a consequence of Theorem 3.13, we have proved

Theorem 3.16. *Let $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring over a field k of characteristic zero, $f_1, \ldots, f_d$ be a system of parameters and $\mathfrak{d} = (f_1, \ldots, f_d)$. Let $R = T/\mathfrak{d}^n$. Then $g(R) \leq \lambda(T/\mathfrak{d}^{n-1})$.*

In order to prove Theorem 3.3, we know need to show that $g(R) \geq \lambda(T/\mathfrak{d}^{n-1})$. We prove this by first computing the trace ideal $\omega^*(\omega)$ of the canonical module and use the fundamental inequalities. We do this by proving the following

Theorem 3.17. *Let $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring over a field k of characteristic zero. Let $f_1, \ldots, f_d$ be a system of parameters and $R = T/(f_1, \ldots, f_d)^n$. Then $\omega^*(\omega) = (f_1, \ldots, f_d)^{n-1}/(f_1, \ldots, f_d)^n$, where ω is the canonical module of R.*

Proof. Let $\mathfrak{d} = (f_1, \ldots, f_d)^n T'$ and $R' \simeq T'/\mathfrak{d}^n$. Let

$$0 \to T'^{b_d} \xrightarrow{\phi} T'^{b_{d-1}} \to \ldots \to T' \to R' \to 0 \qquad (\#)$$

be a minimal resolution of R' over T'. Tensor with R' and define the matrix ψ by the exact sequence $R'^{b_{d+1}} \xrightarrow{\psi} R'^{b_d} \xrightarrow{\phi \otimes R'} R'^{b_{d-1}}$. Then by Lemma 2.21, $\omega^*_{R'}(\omega_{R'})$ is the ideal generated by the entries of the matrix ψ. .

Note that T is free over T' and $R \simeq T \otimes_{T'} R'$. Hence a minimal resolution of R over T is obtained by tensoring (#) by T over T'. Therefore $\omega^*(\omega)$ is the ideal generated by the entries of the matrix $\psi \otimes_{T'} T$. Now, by Theorem 3.4, the ideal in R' generated by the entries of $\psi = \omega^*_{R'}(\omega_{R'}) = \mathfrak{d}^{n-1}/\mathfrak{d}^n$. Therefore, since T is free over T', $\omega^*(\omega) = \mathfrak{d}^{n-1}T/\mathfrak{d}^n T$. □

Proof of Theorem 3.3. By Theorem 3.16, $g(R) \leq \lambda(T/(f_1,\dots,f_d^{n-1}))$. The other inequality follows from Theorem 3.17 which can be seen as follows:

Let ω be the canonical module of R. We know that $g(R) \geq \lambda(R/\omega^*(\omega))$. We get $g(R) \geq \lambda(T/(f_1,\dots,f_d^{n-1}))$ since $\omega^*(\omega) = (f_1,\dots,f_d)^{n-1}/(f_1,\dots,f_d)^n$ and $R = T/(f_1,\dots,f_d)^n$. Thus $g(R) = \lambda(T/(f_1,\dots,f_d^{n-1}))$ proving the theorem. □

Thus, it follows from Theorem 3.16 and Theorem 3.17 that in this case,

$$\lambda(R/\omega^*(\omega)) = \min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\} = g(R).$$

Corollary 3.18. *Let $T = \mathsf{k}[|X_1,\dots,X_d|]$ be a power series ring over a field k of characteristic zero. Let $f_1,\dots,f_d$ be a system of parameters and $R = T/(f_1,\dots,f_d)^n$. Then $g(R) \leq \lambda(R/\operatorname{soc}(R))$.*

Proof. Since

$$(f_1,\dots,f_d)^n :_T (X_1,\dots,X_d) \subseteq (f_1,\dots,f_d)^n :_T (f_1,\dots,f_d) = (f_1,\dots,f_d)^{n-1},$$

we have $\lambda(R/\operatorname{soc}(R)) \geq \lambda(T/(f_1,\dots,f_d^{n-1})) = g(R)$. □

By taking $\mathfrak{d}$ to be the maximal ideal, the following theorem follows immediately from the Theorem 3.3.

Theorem 3.19. *Let* k *be a field of characteristic zero and* $T = \mathsf{k}[|X_1,\dots,X_d|]$ *be a power series ring over* k. *Let* $\mathfrak{m}_T = (X_1,\dots,X_d)$ *be the maximal ideal of* T *and* $R := T/\mathfrak{m}_T^n$. *Then*

$$g(R) = \lambda(T/\mathfrak{m}_T^{n-1}) = \lambda(R/\mathrm{soc}(R)).$$

This theorem also follows immediately from Theorems 3.4 and 3.13.

Remark 3.20. If $R = \mathsf{k}[X_1,\dots,X_d]/(X_1,\dots,X_d)^n$, where k is a field of characteristic zero, it follows from Theorem 3.4 and Theorem 3.13 that $g(R) = \lambda(R/\omega^*(\omega))$. Thus Question 2.19 has a positive answer, i.e., in this case,

$$\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\} = g(R).$$

3.2 Applications

Gorenstein Liaison

Proposition 3.21. *Let* k *be a field of characteristic zero and* $T = \mathsf{k}[|X_1,\dots,X_d|]$ *be a power series ring. Let* $\mathfrak{m} = (X_1,\dots,X_d)$ *and* $\mathfrak{c}_n = (X_1^n,\dots,X_d^n) :_T l^{(d-2)(n-1)}$, *where* $l = (X_1+\cdots+X_d)$. *Then* $\mathfrak{c}_n :_T \mathfrak{m}^n = \mathfrak{m}^{n-1}$.

Proof. Note that $\mathfrak{m}^{n-1}\cdot\mathfrak{m}^n\cdot l^{(d-2)(n-1)} \subseteq \mathfrak{m}^{d(n-1)+1} \subseteq (X_1^n,\dots,X_d^n)$. Hence $\mathfrak{m}^{n-1} \subseteq \mathfrak{c}_n :_T \mathfrak{m}^n$. To prove the other inclusion, let F be a homogeneous form of degree less than $n-1$. Then $F\cdot l^{(d-2)(n-1)}$ is a homogeneous form of degree $d(n-1)-n$ or less. Hence there is some element $G \in \mathfrak{m}^n$ such that $Fl^{(d-2)(n-1)}\cdot G = X_1^{n-1}\cdots X_d^{n-1}$ modulo $(X_1^n,\dots,X_d^n)$ since $T/(X_1^n,\dots,X_d^n)$ is a Gorenstein Artin local ring with socle element $X_1^{n-1}\cdots X_d^{n-1}$. Thus $Fl^{(d-2)(n-1)}\mathfrak{m}^n \not\subseteq (X_1^n,\dots,X_d^n)$ for $F \notin \mathfrak{m}^{n-1}$. Therefore $\mathfrak{c}_n :_T \mathfrak{m}^n \subseteq \mathfrak{m}^{n-1}$, proving the proposition. $\square$

Corollary 3.22. *Let* k *be a field of characteristic zero and* $T = \mathsf{k}[|X_1,\dots,X_d|]$ *be a power series ring. Let* $\mathfrak{m} = (X_1,\dots,X_d)$ *and* $\mathfrak{c}_n = ((X_1^n,\dots,X_d^n) :_T l^s)$*, where* $l = (X_1 + \cdots + X_d)$ *and* $s \geq (d-2)(n-1)-1$*. Then* $(\mathfrak{c}_n :_T \mathfrak{m}^n) = \mathfrak{m}^{(d-1)(n-1)-s}$.

Proof. By taking $S = T/(X_1^n,\dots,X_d^n)$, it follows from Proposition 3.21 that $(\mathfrak{c}_n :_T \mathfrak{m}^n) = \mathfrak{m}^{(d-1)(n-1)-s} + \mathfrak{c}_n$. It remains to prove that $\mathfrak{c}_n \subseteq \mathfrak{m}^{(d-1)(n-1)-s}$.

Let f be a homogeneous element of T such that $f \in \mathfrak{c}$, i.e., $f \cdot l^s \subseteq (X_1^n,\dots,X_d^n)$. Hence by Theorem 3.9, $deg(f) \geq \frac{(d(n-1)-s+1)}{2} \geq (d-1)(n-1)-s$ by the hypothesis on s. This shows that $\mathfrak{c}_n \subseteq m^{(d-1)(n-1)-s}$. □

Let $T = \mathsf{k}[|X_1,\dots,X_d|]$ be a power series ring over a field k. Let $f_1,\dots,f_d$ be a system of parameters. Let $T' = \mathsf{k}[|f_1,\dots,f_d|]$, $\mathfrak{d} = (f_1,\dots,f_d)^n T'$ and $\mathfrak{c}_n = (f_1^n,\dots,f_d^n) :_{T'} l^s$, where $l = f_1 + \cdots + f_d$ and $s \geq (d-2)(n-1)-1$. Since, by Corollary 3.22, $(\mathfrak{c}_n :_{T'} \mathfrak{d}^n) = \mathfrak{d}^{(d-1)(n-1)-s}$ in T', the same holds in T by Remark 3.15. Therefore $(\mathfrak{c}_n T :_T \mathfrak{d}^n T) = \mathfrak{d}^{(d-1)(n-1)-s} T$. Thus we see that

Proposition 3.23. *Let* k *be a field of characteristic zero and* $T = \mathsf{k}[|X_1,\dots,X_d|]$ *be a power series ring. Let* $\mathfrak{d} = (f_1,\dots,f_d)$*, where* $f_1,\dots,f_d$ *form a system of parameters. Let* $l = f_1 + \cdots + f_d$ *and* $s \geq (d-2)(n-1)-1$*. Then* $\mathfrak{c}_n = ((f_1^n,\dots,f_d^n) :_T l^s)$ *is a Gorenstein ideal such that* $(\mathfrak{c}_n :_T \mathfrak{d}^n) = \mathfrak{d}^{(d-1)(n-1)-s}$.

Definition 3.24. *Let* $(T, \mathfrak{m}_T, \mathsf{k})$ *be a regular local ring. An unmixed ideal* $\mathfrak{b} \subseteq T$ *is said to be in the linkage class of a complete intersection (licci) if there is a sequence of ideals* $\mathfrak{c}_n \subseteq \mathfrak{b}_n$*,* $\mathfrak{b}_0 = \mathfrak{b}$*, satisfying*

1) $T/\mathfrak{c}_n$ *is a complete intersection for every* n

2) $\mathfrak{b}_n = \mathfrak{c}_{n-1} :_T \mathfrak{b}_{n-1}$ *and*

3) $\mathfrak{b}_n$ *is a complete intersection for some* n*.*

We say that $\mathfrak{b}$ is *linked* to $\mathfrak{b}_n$ via complete intersections in n steps.

Definition 3.25. *An ideal $\mathfrak{b} \subseteq T$ is said to be in the Gorenstein linkage class of a complete intersection (glicci) if we replace condition (1) above by*

(1) $T/\mathfrak{c}_n$ is Gorenstein for every n.

There are ideals which are glicci but not licci. A result of Huneke and Ulrich in [10] shows that $\mathfrak{m}^n \subseteq \mathsf{k}[X_1, \ldots, X_d]$ is not licci for $d \geq 3$, $n \geq 2$.

Remark 3.26.

1. Let k be a field of characteristic zero and $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring. Let $\mathfrak{d} = (f_1, \ldots, f_d)$, where $f_1, \ldots, f_d$ form a system of parameters. In [16], Kleppe, Migliore, Miró-Roig, Nagel and Peterson show that $\mathfrak{d}^n$ can be linked to $\mathfrak{d}^{n-1}$ via Gorenstein ideals in 2 steps and hence to $\mathfrak{d}$ in $2(n-1)$ steps. But in Proposition 3.23, by taking $s = (d-2)(n-1)$, we see that $\mathfrak{d}^n$ can be linked directly via the Gorenstein ideal $(f_1^n, \ldots, f_d^n) :_{T'} l^{(d-2)(n-1)}$ to $\mathfrak{d}^{n-1}$, and hence to $\mathfrak{d}$, a complete intersection, in $n-1$ steps.

2. In a private conversation, Migliore asked if this technique will show that $\mathfrak{d}^n$ is self-linked. We see that this can be done by taking $s = (d-2)(n-1)-1$ in Proposition 3.23. Thus $\mathfrak{d}^n$ is linked to itself via the Gorenstein ideal $(f_1^n, \ldots, f_d^n) :_{T'} l^{(d-2)(n-1)-1}$.

A Possible Approach to the Glicci Problem

The Glicci problem: Given any homogeneous ideal $\mathfrak{b} \subseteq T := \mathsf{k}[X_1, \ldots, X_d]$, such that $R := T/\mathfrak{b}$ is Cohen-Macaulay, is it true that $\mathfrak{b}$ is glicci?

A possible approach to the glicci problem is the following: Choose $\mathfrak{c}_n \subseteq \mathfrak{b}_n$ to be a Gorenstein ideal which minimizes $\lambda(\mathfrak{b}_n/\mathfrak{c}_n)$. The question is: Does this ensure that $\mathfrak{b}_n$ is a complete intersection for some n?

Example 3.27. Let $T = \mathsf{k}[|X_1, \ldots, X_d|]$, where $\operatorname{char}(\mathsf{k}) = 0$. Let $\mathfrak{d} = (f_1, \ldots, f_d)$ be an ideal generated minimally by a system of parameters. We know that the ideal $\mathfrak{c}_n =$

$(f_1^n,\dots,f_d^n) :_T (f_1+\cdots+f_d)^{(d-2)(n-1)}$ is a Gorenstein ideal closest to $\mathfrak{d}^n$. Now $(\mathfrak{c}_i :_T \mathfrak{d}^i) = \mathfrak{d}^{i-1}$, $2 \le i \le n$, by Proposition 3.23. Thus $\mathfrak{d}^n$ can linked to $\mathfrak{d}$ by choosing a closest Gorenstein ideal at each step.

Compressed Gorenstein Artin Algebras

Definition 3.28. *Let S be a graded Gorenstein Artin quotient of $T = \mathsf{k}[X_1,\dots,X_d]$. We say that S is a compressed Gorenstein algebra of socle degree $t = \operatorname{Max}(S)$, if for each i, $h_S(i)$ is the maximum possible given the embedding dimension (i.e., the minimal number of generators of the maximal ideal) d and socle degree t, i.e., $h_S(i) = \min\{h_T(i), h_T(t-i)\}$ (e.g., see [14]).*

Remark 3.29. When $\operatorname{char}(\mathsf{k}) = 0$, the proofs of Proposition 3.11 and Corollary 3.12 show that $S = T/((X_1^n,\dots,X_d^n) :_T l^{(d-2)(n-1)})$ is a compressed Gorenstein Artin algebra of socle degree $2n-2$. A similar technique also shows that $S = T/((X_1^n,\dots,X_d^n) :_T l^{(d-2)(n-1)-1})$ is a compressed Gorenstein Artin algebra of socle degree $2n-1$.

Remark 3.30.

1. It should be noted from the proof of Proposition 3.11 that S is a graded Gorenstein Artin local ring mapping onto $R = \mathsf{k}[x_1,\dots,x_d]/(x_1,\dots,x_d)^n$ with $\lambda(S)-\lambda(R) = \lambda(R/\operatorname{soc}(R)) = \lambda(\mathsf{k}[x_1,\dots,x_d]/(x_1,\dots,x_d)^{n-1})$ if and only if S is a compressed Gorenstein Artin algebra with embedding dimension d and socle degree $2n-2$. Hence Theorem 3.13 follows from the works of Iarrobino([13]), Fröberg and Laksov([6]) who construct such compressed Gorenstein Artin algebras. We use a different approach to construct one as observed in the next remark, which is what makes it interesting.
2. If one can prove the existence of a compressed Gorenstein Artin k-algebra of embedding dimension d and socle degree $2n-2$, where k is a field of positive characteristic, then the work in Section 1 can be extended to the positive characteristic case.

Remark 3.31.

1. One can also see that if $T = \mathsf{k}[x_1,\ldots,x_d]$ and $S = T/J$ is a compressed Gorenstein Artin algebra of socle degree $2n-2$ and embedding dimension d, then $J :_T (x_1,\ldots,x_d)^n = (x_1,\ldots,x_d)^{n-1}$, by looking at h-vectors. I would like to thank Juan Migliore for pointing this out to me. Once can see this as follows:

Let $\mathfrak{m} = (x_1,\ldots,x_d)$ and $R = T/\mathfrak{m}^n$.

degree i	0	1	2	3	...	n-1	(2n-2) - (n-2)	...	2n-3	2n-2
$h_R(i)$	1	d	$\binom{d+1}{2}$	$\binom{d+2}{3}$	...	$\binom{d+n-2}{n-1}$	0	...	0	0
$h_S(i)$	1	d	$\binom{d+1}{2}$	$\binom{d+2}{3}$	...	$\binom{d+n-2}{n-1}$	$\binom{d+n-3}{n-2}$	...	d	1
$h_{I/J}$	0	0	0	0	...	0	$\binom{d+n-3}{n-2}$	...	d	1

Since T/J is Gorenstein Artin, $I = J :_T (J :_T I)$ and hence I/J is isomorphic to the canonical module of $T/(J :_T I)$. Thus $h_{I/J}(i) = (0,\ldots,0,,\binom{d+n-3}{n-2},\ldots,d,1)$ forces $h_{T/(J:_T I)}(i) = (1,d,\ldots,\binom{d+n-3}{n-2})$. Therefore $J :_T I = \mathfrak{m}^{n-1}$.

2. Similarly, if $S = T/J$ is a compressed Gorenstein Artin algebra of socle degree $2n-1$ and embedding dimension d, it can be seen that $J :_T \mathfrak{m}^n = \mathfrak{m}^n$.

3.3 The Codimension Two Case

We begin this section with the following result of Serre which states that every Gorenstein ideal of codimension two must be a complete intersection ideal.

Remark 3.32. Let $(T,\mathfrak{m}_T,\mathsf{k})$ be a regular local ring of dimension two. Let $\mathfrak{c}$ be an $\mathfrak{m}_T$ primary ideal such that $S = T/\mathfrak{c}$ is a Gorenstein Artin local ring. Then S is a complete intersection ring, i.e., $\mathfrak{c}$ is generated by 2 elements.

To see this, let $0 \to T^{b_1-1} \to T^{b_1} \to T \to S \to 0$ be a minimal resolution of S over T. Then b_1 is the minimal number of generators of $\mathfrak{c}$. Since S is Gorenstein, $\mathrm{soc}(S)$ is a 1-dimensional k-vector space, i.e., $\dim(\mathrm{soc}(S)) = b_1 - 1 = 1$. Thus $b_1 = 2$, proving that S is a complete intersection ring.

Thus, in this case, every Gorenstein $\mathfrak{m}_T$-primary ideal is generated by a regular system of parameters.

Notation: For the rest of this section, we will use the following notation: Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension 2, where that k is infinite. By $e_0(\_)$, we denote the multiplicity of an $\mathfrak{m}_T$-primary ideal in T. For an ideal $\mathfrak{b}$ in T, by $\mathfrak{b}^-$, we denote the integral closure of $\mathfrak{b}$ in T.

Remark 3.33. We state the basic facts needed in this section in this remark. Their proofs can be found in [21] (Chapter 14).

1. Let $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. We define the order of $\mathfrak{b}$ as $\mathrm{ord}(\mathfrak{b}) = \max\{i : \mathfrak{b} \subseteq \mathfrak{m}_T^i\}$.

   Since $\mathfrak{m}_T$ is integrally closed, $\mathrm{ord}(\mathfrak{b}) = \mathrm{ord}(\mathfrak{b}^-)$.

2. Let $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. Since k is infinite, a minimal reduction of $\mathfrak{b}$ is generated by 2 elements.

   Further, if $\mathfrak{c}$ is a minimal reduction of $\mathfrak{b}$, the multiplicity of $\mathfrak{b}$, $e_0(\mathfrak{b}) = \lambda(T/\mathfrak{c})$.

3. The product of integrally closed $\mathfrak{m}_T$-primary ideals is integrally closed. In particular, if $\mathfrak{b}$ is an integrally closed $\mathfrak{m}_T$-primary ideal, then so is $\mathfrak{b}^n$ for each $n \geq 2$.

4. For an $\mathfrak{m}_T$-primary ideal $\mathfrak{b}$, $\lambda((\mathfrak{b} : \mathfrak{m}_T)/\mathfrak{b}) = \mu(\mathfrak{b}) - 1 \leq \mathrm{ord}(\mathfrak{b})$. Further, if $\mathfrak{b}$ is integrally closed, $\mu(\mathfrak{b}) - 1 = \mathrm{ord}(\mathfrak{b})$.

   In particular, this yields $\mu(\mathfrak{b}) \leq \mu(\mathfrak{b}^-)$.

Proposition 3.34. *Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension two and let $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. The closest (in terms of length) Gorenstein ideals contained in $\mathfrak{b}$ are its minimal reductions.*

Proof. Let $\mathfrak{c} \subseteq \mathfrak{b}$ be any Gorenstein ideal (and hence a complete intersection by the above remark). It is easy to see that $\lambda(T/\mathfrak{c}) \geq \lambda(T/(f,g))$, where $(f,g) \subseteq \mathfrak{b}$ is a minimal reduction of $\mathfrak{b}$. The reason is that

$$\begin{aligned} \lambda(T/\mathfrak{c}) &= e_0(\mathfrak{c}) \\ &\geq e_0(\mathfrak{b}) && \text{since } \mathfrak{c} \subseteq \mathfrak{b} \\ &= \lambda(T/(f,g)). \end{aligned}$$

As a consequence,

$$\lambda(T/\mathfrak{c}) - \lambda(T/\mathfrak{b}) \geq \lambda(T/(f,g)) - \lambda(T/\mathfrak{b}),$$

$$\text{i.e. } \lambda(\mathfrak{b}/\mathfrak{c}) \geq \lambda(\mathfrak{b}/(f,g)).$$

Thus the closest Gorenstein ideal contained in $\mathfrak{b}$ is a minimal reduction (f,g). $\square$

We now prove the following theorem which shows that $g(R) \leq \lambda(R/\mathrm{soc}(R))$ where R is the Artinian quotient of a 2-dimensional regular local ring.

Theorem 3.35. *Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension 2, with infinite residue field k. Set $R = T/\mathfrak{b}$ where $\mathfrak{b}$ is an $\mathfrak{m}_T$-primary ideal. Then $g(R) \leq \lambda(R/\mathrm{soc}(R))$, i.e. there is a Gorenstein ring S mapping onto R such that $\lambda(S) - \lambda(R) \leq \lambda(R/\mathrm{soc}(R))$.*

In order to prove Theorem 3.35, we use a couple of formulae for $e_0(\mathfrak{b})$ and $\lambda(R)$ (which can be found, for example, in [15]). We need the following notation.

Let $(T, \mathfrak{m})$ and $(T', \mathfrak{n})$ be two-dimensional regular local rings. We say that T' birationally dominates T if $T \subseteq T'$, $\mathfrak{n} \cap T = \mathfrak{m}$ and T and T' have the same quotient field. We denote this by $T \leq T'$. Let $[T' : T]$ denote the degree of the field extension $T/\mathfrak{m} \subseteq T'/\mathfrak{n}$.

Further if $\mathfrak{b}$ is an $\mathfrak{m}$-primary ideal in T, let $\mathfrak{b}^{T'}$ be the ideal in T' obtained from $\mathfrak{b}$ by factoring $\mathfrak{b}T' = x^{\mathrm{ord}(\mathfrak{b})}\mathfrak{b}^{T'}$. The following theorem ([15], Theorem 3.7) gives a formula for $e_0(\mathfrak{b})$.

Theorem 3.36 (Multiplicity Formula)**.** *Let* $(T, \mathfrak{m}_T, \mathsf{k})$ *be a two-dimensional regular local ring and* $\mathfrak{b}$ *be an* $\mathfrak{m}_T$*-primary ideal. Then*

$$e_0(\mathfrak{b}) = \sum_{T \leq T'} [T' : T] \mathrm{ord}(\mathfrak{b}^{T'})^2.$$

The following formula ([15], Theorem 3.10) is attributed to Hoskin and Deligne. For historical remarks regarding the various proofs of this formula, one can refer [15].

Theorem 3.37 (Hoskin-Deligne Formula)**.** *Let* T*,* $\mathfrak{b}$ *and* R *be as in Theorem 3.35. Further assume that* $\mathfrak{b}$ *is an integrally closed ideal. Then,*

$$\lambda(R) = \sum_{T \leq T'} \binom{\mathrm{ord}(\mathfrak{b}^{T'}) + 1}{2} [T' : T].$$

Corollary 3.38. *Let* T*,* $\mathfrak{b}$ *and* R *be as in the Hoskin-Deligne formula. Then we have the inequality*

$$e_0(\mathfrak{b}) + \mathrm{ord}(\mathfrak{b}) \leq 2\lambda(R).$$

Proof. By Theorem 3.36, we have $e_0(\mathfrak{b}) = \sum_{T \leq T'} \mathrm{ord}(\mathfrak{b}^{T'})^2 [T' : T]$.

Using the Hoskin-Deligne formula, we see that

$$\lambda(R) = \sum_{T \leq T'} \frac{\mathrm{ord}(\mathfrak{b}^{T'})^2 + \mathrm{ord}(\mathfrak{b}^{T'})}{2} [T' : T]$$

giving us

$$2\lambda(R) = e_0(\mathfrak{b}) + \sum_{T \leq T'} \mathrm{ord}(\mathfrak{b}^{T'}) [T' : T].$$

Since $T \leq T$ and $\mathfrak{b}^T = \mathfrak{b}$, we get the required inequality. □

Corollary 3.39. *Let T, R and $\mathfrak{b}$ be as in Theorem 3.35. Then*

$$e_0(\mathfrak{b}) + \mu(\mathfrak{b}) - 1 \leq 2\lambda(T/\mathfrak{b}).$$

Proof: Let $\mathfrak{b}^-$ be the integral closure of $\mathfrak{b}$. By the previous corollary, we have $e_0(\mathfrak{b}^-) + \operatorname{ord}(\mathfrak{b}^-) \leq 2\lambda(T/\mathfrak{b}^-)$. Since $\mathfrak{b}^-$ is integrally closed, $\operatorname{ord}(\mathfrak{b}^-) = \mu(\mathfrak{b}^-) - 1$. Thus we get $e_0(\mathfrak{b}^-) + \mu(\mathfrak{b}^-) - 1 \leq 2\lambda(T/\mathfrak{b}^-)$. Now $e_0(\mathfrak{b}) = e_0(\mathfrak{b}^-)$, $\mu(\mathfrak{b}) \leq \mu(\mathfrak{b}^-)$ and $\lambda(T/\mathfrak{b}^-) \leq \lambda(T/\mathfrak{b})$, giving the required inequailty. $\square$

Proof of Theorem 3.35. For any ideal $\mathfrak{b}$ in T, we have $\mu(\mathfrak{b}) - 1 = \lambda((\mathfrak{b} : \mathfrak{m})/\mathfrak{m})$. But $(\mathfrak{b} : \mathfrak{m})/\mathfrak{b} \simeq \operatorname{soc}(R)$. Thus by the previous corollary, we have

$$e_0(\mathfrak{b}) + \lambda(\operatorname{soc}(R)) \leq 2\lambda(R). \qquad (*)$$

Let (f, g) be a minimal reduction of $\mathfrak{b}$. Then $S := T/\mathfrak{b}$ is a complete intersection ring (and hence Gorenstein) mapping onto R. Moreover $\lambda(S) = e_0(\mathfrak{b})$. Thus $(*)$ can be read as $\lambda(S) + \lambda(\operatorname{soc}(R)) \leq 2\lambda(R)$. Rearranging, we get $\lambda(S) - \lambda(R) \leq \lambda(R) - \lambda(\operatorname{soc}(R))$. This proves that $g(R) \leq \lambda(R/\operatorname{soc}(R))$. $\square$

Chapter 4

Fibre Products and Connected Sums

4.1 Fibre Products of Noetherian Local Rings

Definition 4.1. *Let $(R_1, \mathfrak{m}_1, \mathsf{k})$ and $(R_2, \mathfrak{m}_2, \mathsf{k})$ be two Noetherian local rings mapping onto R via the maps π_1 and π_2 respectively. Let* $\ker(\pi_i) = I_i$.

We define the fibre product of R_1 and R_2 over R, denoted $R_1 \times_R R_2$, as $\{(r_1, r_2) \in R_1 \times R_2 : \pi_1(r_1) = \pi_2(r_2)\}$.

The fibre product $R_1 \times_R R_2$ is a ring under component-wise addition and multiplication.

The following are some basic properties which follow immediately from the definition of the fibre product.

Remark 4.2. Let $(R_i, \mathfrak{m}_i, \mathsf{k})$, $i = 1, 2$ and $(R, \mathfrak{m}, \mathsf{k})$ be Noetherian local rings with surjective ring homomorphisms $\pi_i : R_i \longrightarrow R$. Write $R := R_1/I_1 \simeq R_2/I_2$ for ideals I_1 and I_2 in R_1 and R_2 respectively.

1. The ring $R_1 \times_R R_2$ is also local with unique maximal ideal $\mathfrak{m} = \{(m_1, m_2) \in \mathfrak{m}_1 \times \mathfrak{m}_2 : \pi_1(m_1) = \pi_2(m_2)\}$ and residue field k.
2. We have $R_1 \times_R R_2 \subseteq R_1 \times_{\mathsf{k}} R_2 \subseteq R_1 \times R_2$.

3. Let $(r_1,r_2) \in R_1 \times_R R_2$. Then we have (a) $r_1 \in I_1$ if and only if $r_2 \in I_2$ and (b) r_1 is a unit in R_1 if and only if r_2 is a unit in R_2 which are both equivalent to (r_1,r_2) being unit in $R_1 \times_R R_2$.

4. One can see that if $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$, the natural projection maps from $R_1 \times_R R_2$ to R_1 and R_2 induce the isomorphisms $R_1/J_1 \simeq (R_1 \times_R R_2)/(J_1,I_2)$ and $R_2/J_2 \simeq (R_1 \times_R R_2)/(I_1,J_2)$.

In particular, we have

$$R_1 \simeq (R_1 \times_R R_2)/(0,I_2), \quad R_2 \simeq (R_1 \times_R R_2)/(I_1,0) \text{ and } \quad R \simeq (R_1 \times_R R_2)/I,$$

where $I = \{(r_1,r_2) \in R_1 \times_R R_2 : r_1 \in I_1, r_2 \in I_2\}$.

Thus if $R_2 = R$, then $R_1 \times_R R \simeq R_1$.

5. Let I be as in (4). If $0 :_{R_i} I_i \subseteq I_i$, then $(0 :_{R_1 \times_R R_2} I) = \{(r_1,r_2) : r_i \in (0 :_{R_i} I_i)\}$.
In particular, taking $I_i = \mathfrak{m}_i$, we get $\operatorname{soc}(R_1 \times_{\mathsf{k}} R_2) = \{(r_1,r_2) : r_i \in \operatorname{soc}(R_i)\}$.

6. If we further assume that $(R_1,\mathfrak{m}_1,\mathsf{k})$ and $(R_2,\mathfrak{m}_2,\mathsf{k})$ are Artinian local rings, (4) yields

$$\lambda(R_1 \times_R R_2) = \lambda(R_1) + \lambda(I_2) = \lambda(R_1) + \lambda(R_2) - \lambda(R).$$

Moreover, by (5), if neither R_1 nor R_2 is isomorphic to k, then $R_1 \times_{\mathsf{k}} R_2$ cannot be Gorenstein.

Proposition 4.3. *Let $S \xrightarrow{p_1} R_1 \xrightarrow{\pi_1} R$ and $S \xrightarrow{p_2} R_2 \xrightarrow{\pi_2} R$ be such that $\pi_1 p_1 = \pi_2 p_2$. Then there is a ring homomorphism $\phi : S \longrightarrow R_1 \times_R R_2$ defined by $\phi(s) = (p_1(s), p_2(s))$. Furthermore,*

a) *if* $\ker(p_1) \cap \ker(p_2) = 0$*, then ϕ is injective and*

b) *if* $\ker(\pi_i p_i) = \ker(p_1) + \ker(p_2)$*, then ϕ is surjective.*

Proof. Since $\pi_1 p_1(s) = \pi_2 p_2(s)$, for every $s \in S$, we have $(p_1(s), p_2(s)) \in R_1 \times_R R_2$. Thus the map $\phi : S \longrightarrow R_1 \times_R R_2$ defined as $\phi(s) = (p_1(s), p_2(s))$ is well-defined. Moreover, since p_1 and p_2 are ring homomorphisms, so is ϕ.

a) Note that $\ker(\phi) = \{s \in S : p_1(s) = 0 = p_2(s)\}$. Hence $\ker(\phi) = \ker(p_1) \cap \ker(p_2)$. Thus ϕ is injective if $\ker(p_1) \cap \ker(p_2) = 0$.

b) Let $(r_1, r_2) \in R_1 \times_R R_2$. Then there are $s_1, s_2 \in S$, such that $p_1(s_1) = r_1$ and $p_2(s_2) = r_2$. Since $(r_1, r_2) \in R_1 \times_R R_2$, we see that $\pi_1 p_1(s_1) = \pi_2 p_2(s_2) = \pi_1 p_1(s_2)$. Hence $s_1 - s_2 \in \ker(\pi_1 p_1)$.

Since $\ker(\pi_1 p_1) = \ker(p_1) + \ker(p_2)$, there are elements $x \in \ker(p_1)$ and $y \in \ker(p_2)$ such that $s_1 - s_2 = x - y$. Set $s = s_1 + x = s_2 + y$. Then $p_1(s) = r_1$ and $p_2(s) = r_2$. Thus $\phi(s) = (r_1, r_2)$, which proves that ϕ is surjective. □

Corollary 4.4. *Let $(R', \mathfrak{m}, \mathsf{k})$ be an Artinian local ring, I_1 and I_2 be ideals in R'. Then $R'/(I_1 \cap I_2) \simeq R'/I_1 \times_{R'/(I_1+I_2)} R'/I_2$.*

In particular, if $I_1 \cap I_2 = 0$, then $R' \simeq R'/I_1 \times_{R'/(I_1+I_2)} R'/I_2$.

Proof. The proof follows from Proposition 4.3 by setting $S = R'/(I_1 \cap I_2)$, $R_1 = R'/I_1$, $R_2 = R'/I_2$ and $R = R'/(I_1 + I_2)$. Note that $\ker(p_i) = I_i/(I_1 \cap I_2)$, $i = 1, 2$ and $\ker(\pi_i p_i) = (I_1 + I_2)/(I_1 \cap I_2)$. □

Proposition 4.5. *Let $S_1 \overset{p_1}{\twoheadrightarrow} R_1 \overset{\pi_1}{\twoheadrightarrow} R$ and $S_2 \overset{p_2}{\twoheadrightarrow} R_2 \overset{\pi_2}{\twoheadrightarrow} R$ be surjective ring homomorphisms. Then $\pi : S_1 \times_R S_2 \longrightarrow R_1 \times_R R_2$ given by $\pi(s_1, s_2) = (p_1(s_1), p_2(s_2))$ is a surjective ring homomorphism.*

Proof. Clearly π is well-defined. Since the addition and multiplication are defined component-wise on both $S_1 \times_R S_2$ and $R_1 \times_R R_2$, it is easily verified that π is a ring homomorphism. It remains to check that π is surjective.

Let $(r_1, r_2) \in R_1 \times_R R_2$. There exist $s_1 \in S_1$ and $s_2 \in S_2$ such that $p_1(s_1) = r_1$ and $p_2(s_2) = r_2$. Since $\pi_1(r_1) = \pi_2(r_2)$, $(\pi_1 \circ p_1)(s_1) = (\pi_2 \circ p_2)(s_2)$, i.e., $(s_1, s_2) \in S_1 \times_R S_2$. By definition of π, $\pi(s_1, s_2) = (r_1, r_2)$. Thus π is surjective. $\square$

4.2 Connected Sums of Gorenstein Artin Local Rings

The notion of a connected sum over k of two Gorenstein Artin local rings was introduced to me by L. Avramov and F. W. Moore in a private communication. In this section, we present the notion of a connected sum over an arbitrary Gorenstein Artin quotient of two Gorenstein Artin local rings and study its properties.

Certain hypotheses are necessary for the key definition and most results in this section. We record them as follows:

Notation:

1. We say that the triple (S, A, R) satisfies the *connected sum hypotheses* (CH) if S and R are Gorenstein Artin local rings and A is an Artinian local ring such that:

i) $R = S/J$, $A = S/I$ for ideals $I \subseteq J$ in S and

ii) $0 :_S J \subseteq I$.

Since S is Gorenstein and maps onto R, $\omega_R \simeq 0 :_S J$. Since R is Gorenstein, ω_R is cyclic. Write $0 :_S J = (w)S$. Thus (ii) can be rephrased as

ii)$'$ $w \in I$.

2. We say that the pair (S, R) satisfies (CH) if the triple (S, R, R) satisfies (CH), i.e., S and R are Gorenstein Artin local rings such that $R = S/J$ and $0 :_S J \subseteq J$.

The following example illustrates some cases where the above hypotheses hold.

Example 4.6.

1. Let $(S,\mathfrak{m},\mathsf{k})$ be a Gorenstein Artin local ring that is not a field. Since $\mathsf{k} = S/\mathfrak{m}$ and $0 :_S \mathfrak{m} = \operatorname{soc}(S) \subseteq \mathfrak{m}$, we see that the pair (S,k) satisfies (CH).

2. Suppose the triple (S,A,R) satisfies (CH). Then so does the pair (S,R).

3. Suppose the pair (S,R) satisfies (CH). Write $R = S/J$. Then $(w) = 0 :_S J \subseteq J$. Let $I \subseteq S$ be an ideal such that $w \in I \subseteq J$. Then the triple $(S,S/I,R)$ satisfies (CH).

4. (= (1) + (3)) In particular, if $(S,\mathfrak{m},\mathsf{k})$ is a Gorenstein Artin local ring that is not a field and $I \subseteq S$ is a non-zero ideal, then the triple $(S,S/I,\mathsf{k})$ satisfies (CH), since $(0 :_S \mathfrak{m}) = \operatorname{soc}(S)$ is contained in every non-zero ideal of S.

Setup 4.7. We will use the following setup in this section: (S_i,R_i,R), $i = 1,2$, satisfy (CH). Write $R_i = S_i/I_i, R = S_i/J_i$ and $(w_i) = 0 :_{S_i} J_i$. Then $w_i \in I_i \subseteq J_i$ and hence $(w_1,-w_2) \in S_1 \times_R S_2$.

Note that since each S_i is Gorenstein, $(0 :_{S_i} w_i) = J_i$.

Definition 4.8. *Suppose (S_1,R) and (S_2,R) satisfy (CH). With notations as in Setup 4.7, we define a connected sum of S_1 and S_2 over R, denoted $S_1\#_R S_2$, as $(S_1 \times_R S_2)/(S_1 \times_R S_2)(w_1,-w_2)$.*

Note: The definition of a connected sum depends on the generator w_i of $0 :_{S_i} I_i$ chosen. Hence a connected sum is unique up to a unit in $S_1 \times S_2$.

The following lemma helps us compute the length of a connected sum of two Gorenstein Artin local rings.

Lemma 4.9. *Suppose (S_1,R) and (S_2,R) satisfy (CH). With notations as in Setup 4.7, $\lambda((w_1,-w_2)(S_1 \times_R S_2)) = \lambda(R)$.*

Proof. We will first show that $(w_1,-w_2)(S_1 \times_R S_2) \simeq (w_1)S_1$. Let $\pi_i : S_i \twoheadrightarrow R$ be the natural surjective maps and $W = (w_1,-w_2)(S_1 \times_R S_2)$. Consider the map $\phi : W \longrightarrow$

$(w_1)S_1$ given by $(s_1,s_2)(w_1,-w_2) \mapsto s_1w_1$. Given $s_1 \in S_1$, there is an element $s_2 \in S_2$ such that $\pi_1(s_1) = \pi_2(s_2)$ in R. Hence $(s_1,s_2) \in S_1 \times_R S_2$, thus $\phi((s_1,s_2)(w_1,-w_2)) = s_1w_1$ which proves that ϕ is onto.

Now suppose $\phi((s_1,s_2)(w_1,-w_2)) = 0$. Then $s_1w_1 = 0$, i.e., $s_1 \in 0:_{S_1} w_1 = J_1$. By Remark 4.2(3), this forces $s_2 \in J_2 = 0:_{S_2} w_2$. Thus $s_2w_2 = 0$, i.e., $(s_1,s_2)(w_1,-w_2) = 0$ in W proving the injectivity of ϕ.

This proves that ϕ is an isomorphism. Since $(w_1)S_1 \simeq \omega_R$ with notations and assumptions as in Setup 4.7, we see that $(w_1,-w_2)(S_1 \times_R S_2) \simeq \omega_R$ proving the lemma. $\square$

As a consequence of Lemma 4.9, we see that $\lambda(S_1 \#_R S_2) = \lambda(S_1 \times_{\mathsf{k}} S_2) - \lambda(R)$. Thus we have

$$\lambda(S_1 \#_R S_2) = \lambda(S_1) + \lambda(S_2) - 2\lambda(R).$$

The following theorem is why we are interested in studying connected sums of two Gorenstein Artin local rings.

Theorem 4.10. *Let S_1 and S_2 be two Gorenstein Artin local rings, neither of which is isomorphic to its residue field. Let $R = S_i/J_i$ be a Gorenstein Artin quotient of S_i, $i = 1,2$ such that $(w_i) = (0:_{S_i} J_i) \subseteq J_i$, i.e., the pairs (S_i,R) satisfies (CH). Then $S_1 \#_R S_2$ is a Gorenstein Artin local ring.*

We need the following lemmas in our proof of Theorem 4.10.

Lemma 4.11. *Suppose (S_1,R) and (S_2,R) satisfy (CH). Let $(s_1,s_2) \in S_1 \times_R S_2$ be such that its image in $S_1 \#_R S_2$ is in $\operatorname{soc}(S_1 \#_R S_2)$. Then $s_i \in \mathfrak{m}_i$.*

Proof. Let the notations be as in Setup 4.7 and let π be the natural projection from $S_1 \times_R S_2$ onto $S_1 \#_R S_2$. Suppose s_1 is a unit in S_1. Then by Remark 4.2(3), (s_1,s_2) is a unit in $S_1 \times_R S_2$. Since $\pi(s_1,s_2) \in \operatorname{soc}(S_1 \#_R S_2)$ is also a unit, $S_1 \#_R S_2 \simeq \mathsf{k}$. This shows that $(S_1 \times_R S_2)(w_1, -w_2)$ is the unique maximal ideal in $S_1 \times_R S_2$. Thus (w_i) is the maximal ideal of S_i. Since $w_i \in J_i \subseteq \mathfrak{m}_i, R \simeq S_i/J_i \simeq \mathsf{k}$. Now $J_i \simeq \mathfrak{m}_i$ implies that $(w_i) = (0 :_{S_i} \mathfrak{m}_i) = \operatorname{soc}(S_i)$. Hence $\lambda(\mathfrak{m}_i) = 1$. Thus $\lambda(S_1 \#_R S_2) = 2$ which contradicts the fact that $\lambda(S_1 \#_R S_2) \simeq \mathsf{k}$.

Therefore $s_1 \in \mathfrak{m}_1$. A similar argument shows that $s_2 \in \mathfrak{m}_2$. $\square$

Lemma 4.12. *Suppose (S_1, R) and (S_2, R) satisfy (CH). With notations as in Setup 4.7,* $\operatorname{soc}(S_1 \times_R S_2) \cap (S_1 \times_R S_2)(w_1, -w_2) \neq 0$, *where* $(w_i) = (0 :_{S_i} J_i)$.

Proof. Since S_1 is Gorenstein, there is an element $u_1 \in S_1$ such that $(u_1 w_1) = \operatorname{soc}(S_1)$. Let $\pi_i : S_i \longrightarrow R$ be the natural projection maps. Choose $u_2 \in S_2$ such that $\pi_1(u_1) = \pi_2(u_2)$.

We will now show that $u_2 w_2 \in \operatorname{soc}(S_2)$. Let $x_2 \in \mathfrak{m}_2$. Choose $x_1 \in \mathfrak{m}_1$ such that $\pi_1(x_1) = \pi_2(x_2)$. Since $(u_1 w_1) x_1 = 0, u_1 x_1 \in J_1$. This forces $u_2 x_2 \in J_2$ by Remark 4.2(3). Hence $(u_2 x_2) w_2 = 0$, i.e., $u_2 w_2 \in \operatorname{soc}(S_2)$.

Since $(u_1, u_2) \in S_1 \times_R S_2$ and $\operatorname{soc}(S_1 \times_R S_2) = \operatorname{soc}(S_1) \oplus \operatorname{soc}(S_2), 0 \neq (u_1 w_1, -u_2 w_2) \in \operatorname{soc}(S_1 \times_R S_2) \cap (S_1 \times_R S_2)(w_1, -w_2)$. $\square$

Proof of Theorem 4.10. We will prove that $\operatorname{soc}(S_1 \times_R S_2)$ maps onto $\operatorname{soc}(S_1 \#_R S_2)$. The facts that $\operatorname{soc}(S_1 \#_R S_2) \neq 0$, $\lambda(\operatorname{soc}(S_1 \times_R S_2) = 2$ and $\operatorname{soc}(S_1 \times_R S_2) \cap (S_1 \times_R S_2)(w_1, -w_2) \neq 0$, force $\lambda(\operatorname{soc}(S_1 \#_R S_2)) = 1$ proving the theorem.

Let π be the natural projection from $S_1 \times_R S_2$ onto $S_1 \#_R S_2$. Let $(s_1, s_2) \in S_1 \times_R S_2$ be such that $\pi(s_1, s_2) \in \operatorname{soc}(S_1 \#_R S_2)$. We want to show that $(s_1, s_2) \in \operatorname{soc}(S_1 \times_R S_2) + (S_1 \times_R S_2)(w_1, -w_2)$.

<u>Claim 1</u>: $s_i \in w_i S_i$, $i = 1, 2$.

By Lemma 4.11, $s_i \in \mathfrak{m}_i$. Let $(t_i) = \operatorname{soc}(S_i)$. For every $j_1 \in J_1$, there is an element $(u_1, u_2) \in S_1 \times_R S_2$ such that $(s_1, s_2)(j_1, t_2) = (u_1, u_2)(w_1, -w_2)$. Now $s_2 t_2 = 0$ yields

$u_2w_2 = 0$, i.e., $u_2 \in J_2$. By Remark 4.2(3), this happens if and only if $u_1 \in J_1$ which forces $s_1 j_1 = 0$. Thus $s_1 \in (0 :_{S_1} J_1) = w_1S_1$. Similarly, we can show that $s_2 \in w_2S_2$, proving Claim 1. Write $s_1 = r_1w_1$ and $s_2 = r_2w_2$ for some $r_1 \in S_1$ and $r_2 \in S_2$. Note that (r_1, r_2) need not be in $S_1 \times_R S_2$. However, we do have the following:

Claim 2: $(\pi_1(r_1) + \pi_2(r_2)) \in \operatorname{soc}(R)$.

Let $x \in \pi_i(\mathfrak{m}_i)$, which is the unique maximal ideal of R. Choose $x_i \in \mathfrak{m}_i$ such that $\pi_i(x_i) = x$. Since $\pi(r_1w_1, r_2w_2) \in \operatorname{soc}(S_1\#_RS_2)$, there exist $t_1 \in S_1$ and $t_2 \in S_2$ satisfying $\pi_1(t_1) = \pi_2(t_2)$ such that $(r_1w_1, r_2w_2)(x_1, x_2) = (t_1, t_2)(w_1, -w_2)$. Thus $(r_1x_1 - t_1)w_1 = 0$ and $(r_2x_2 + t_2)w_2 = 0$ which forces $r_1x_1 - t_1 \in 0 :_{S_1} w_1 = I_1$ and $r_2x_2 + t_2 \in 0 :_{S_2} w_2 = I_2$. Thus $\pi_1(r_1x_1) = \pi_1(t_1) = \pi_2(t_2) = -\pi_2(r_2x_2)$, i.e., $(\pi_1(r_1) + \pi_2(r_2))x = 0$. Since x is an arbitrary element in the maximal ideal of R, Claim 2 is proved.

Choose $v_i \in S_i$ such that $\pi_i(v_i) = \pi_1(r_1) + \pi_2(r_2)$. Then $\pi_1(r_1) = \pi_2(v_2 - r_2)$. i.e., $(r_1, v_2 - r_2) \in S_1 \times_R S_2$. Rewrite

$$(s_1, s_2) = (r_1w_1, r_2w_2) = (r_1, v_2 - r_2)(w_1, -w_2) + (0, v_2)(w_1, w_2). \qquad (*)$$

Claim 3: $v_2w_2 \in \operatorname{soc}(S_2)$. Let $x_2 \in \mathfrak{m}_2$. Then $\pi_2(v_2x_2) = 0$ since $\pi_2(v_2)\operatorname{soc}(R)$. Thus $v_2x_2 \in J_2 = (0 :_{S-2} w_2)$. Thus $(v_2w_2)x_2 = 0$, which gives us $v_2w_2 \in \operatorname{soc}(S_2)$ proving Claim 3.

As a consequence of Claim 3, we see that $(0, v_2)(w_1, w_2) \in \operatorname{soc}(S_1 \times_R S_2)$. Thus by $(*)$, $(s_1, s_2) \in (S_1 \times_R S_2)(w_1, -w_2) + \operatorname{soc}(S_1 \times_R S_2)$, proving the proposition. $\square$

Since a connected sum is a Gorenstein Artin local ring, one can use this notion in the study of Gorenstein colength. We first see some properties of connected sums and then use them to give bounds on the Gorenstein colength of fibre products of Artinian rings.

Remark 4.13.

1. By Remark 4.2(4), since $(w_i) \subseteq J_i$, we see that the map $(S_1 \times_R S_2) \twoheadrightarrow S_i/(w_i)$ factors through $S_1\#_R S_2$. Thus $S_1/(w_1) \simeq (S_1\#_R S_2)/(w_1 S_1, J_1)(S_1\#_R S_2)$.

2. Let $I_i \subseteq S_i$ be an ideal such that $w_i \in I_i \subseteq J_i$. The above map can further be composed with the natural surjection $S_i/(w_i) \twoheadrightarrow S_i/I_i$ to see that the map $S_1 \times_R S_2 \longrightarrow S_i/I_i$ factors through $S_1\#_R S_2$. Thus $S_1/I_1 \simeq (S_1\#_R S_2)/(I_1, J_2)(S_1\#_R S_2)$ and $S_2/I_2 \simeq (S_1\#_R S_2)/(J_1, I_2)(S_1\#_R S_2)$.

Proposition 4.14. *With notation as in Setup 4.7, $S_1\#_R S_2$ maps onto R and $\omega_R \simeq (w_1, w_2)(S_1\#_R S_2)$.*

Moreover, the pair $(S_1\#_R S_2, R)$ satisfies (CH).

Proof. As a particular case in Remark 4.13(2), note that $R \simeq (S_1\#_R S_2)/(J_1, J_2)(S_1\#_R S_2)$ which proves the first part of the proposition. Now, since $S_1\#_R S_2$ is a Gorenstein Artin ring and $R \simeq (S_1\#_R S_2)/(J_1, J_2)(S_1\#_R S_2)$, we have $\omega_R \simeq 0 :_{S_1\#_R S_2} (J_1, J_2)$. Thus we need to prove the following:

<u>Claim</u>: $0 :_{S_1\#_R S_2} (J_1, J_2)(S_1\#_R S_2) = (w_1, w_2)(S_1\#_R S_2)$.

Since $S_1\#_R S_2$ is Gorenstein, it is enough to prove by duality that $(J_1, J_2)(S_1\#_R S_2) = 0 :_{S_1\#_R S_2} (w_1, w_2)(S_1\#_R S_2)$.

Since $0 :_{S_i} J_i = (w_i)$, it is clear that $(J_1, J_2)(S_1\#_R S_2) \subseteq 0 :_{S_1\#_R S_2} (w_1, w_2)(S_1\#_R S_2)$.

To prove the other inclusion, let $(s_1, s_2) \in S_1 \times_R S_2$ be an element whose image in $S_1\#_R S_2$ is in $0 :_{S_1\#_R S_2} (w_1, w_2)(S_1\#_R S_2)$, i.e., $(s_1, s_2)(w_1, w_2) \in (w_1, -w_2)(S_1 \times_R S_2)$. Let $(s_1, s_2)(w_1, w_2) = (c_1, c_2)(w_1, -w_2)$ for some $(c_1, c_2) \in S_1 \times_R S_2$. Note that $\pi_1(s_1) = \pi_2(s_2)$ and $\pi_1(c_1) = \pi_2(c_2)$, where $\pi_i : S_i \longrightarrow R$ are the natural projections.

Thus we see that $(s_1 - c_1)w_1 = 0$ in S_1 and $(s_2 + c_2)w_2 = 0$ in S_2, which forces $s_1 - c_1 \in 0 :_{S_1} w_1 = J_1$ and $s_2 + c_2 \in 0 :_{S_2} w_2 = J_2$. Hence $\pi_1(s_1 - c_1) = 0$ and $\pi_2(s_2 +$

$c_2) = 0$. Therefore $\pi_2(c_2) = \pi_1(c_1) = \pi_1(s_1) = \pi_2(s_2) = -\pi_2(c_2)$ which forces each of the terms to be zero. Thus $\pi_i(s_i) = 0$ which gives $s_i \in J_i$, proving the claim.

Since $R \simeq (S_1 \#_R S_2)/(J_1, J_2)(S_1 \#_R S_2)$, $0 :_{S_1 \#_R S_2} (J_1, J_2)(S_1 \#_R S_2) = (w_1, w_2)(S_1 \#_R S_2)$ and $(w_1, w_2) \in (J_1, J_2)$, the pair $(S_1 \#_R S_2, R)$ satisfies (CH). $\square$

Proposition 4.15. *Suppose (S_1, R_1, R) and (S_2, R_2, R) satisfy (CH). Then the triple $(S_1 \#_R S_2, R_1 \times_R R_2, R)$ satisfies (CH). Thus*

$$g(R_1 \times_R R_2) \leq [\lambda(S_1) - \lambda(R_1)] + [\lambda(S_2) - \lambda(R_2)] - \lambda(R).$$

Proof. Let the notation be as Setup 4.7. By Proposition 4.14, the pair $(S_1 \#_R S_2, R)$ satisfies (CH). Hence in order to prove that $(S_1 \#_R S_2, R_1 \times_R R_2, R)$ satisfies (CH), it is enough to prove that $S_1 \#_R S_2$ maps onto $R_1 \times_R R_2$ and if $R_1 \times_R R_2 \simeq (S_1 \#_R S_2)/I$, then $0 :_{S_1 \#_R S_2} (J_1, J_2)(S_1 \#_R S_2) = (w_1, w_2)(S_1 \#_R S_2) \subseteq I$.

By Proposition 4.5, there is a surjective map $\pi : S_1 \times_R S_2 \twoheadrightarrow R_1 \times_R R_2$, given by $\pi(s_1, s_2) = (\pi_1(s_1), \pi_2(s_2))$ where $\pi_i : S_i \twoheadrightarrow R_i$ are the natural projections. Note that since $w_i \in I_i$, $\pi_i(w_i) = 0$. Thus $\pi(w_1, -w_2) = 0$ in $R_1 \times_R R_2$. Therefore π factors through $S_1 \#_R S_2$ and hence $R_1 \times_R R_2 \simeq (S_1 \#_R S_2)/(I_1, I_2)(S_1 \#_R S_2)$. Since $(w_1, w_2)(S_1 \#_R S_2) \subseteq (I_1, I_2)(S_1 \#_R S_2)$, we see that $(S_1 \#_R S_2, R_1 \times_R R_2, R)$ satisfies (CH).

Now since $S_1 \#_R S_2$ is a Gorenstein Artin local ring mapping onto $R_1 \times_R R_2$, we have

$$\begin{aligned} g(R_1 \times_R R_2) &\leq \lambda(S_1 \#_R S_2) - \lambda(R_1 \times_R R_2) \\ &= (\lambda(S_1) + \lambda(S_2) - 2\lambda(R)) - (\lambda(R_1) + \lambda(R_2) - \lambda(R)) \\ &= [\lambda(S_1) - \lambda(R_1)] + [\lambda(S_2) - \lambda(R_2)] - \lambda(R). \end{aligned}$$

$\square$

Corollary 4.16. *If* (S_i, R_i, R), $i = 1, \ldots, d$, *satisfy (CH), then* $(((S_1 \#_R S_2) \#_R S_3) \#_R \cdots) \#_R S_d$ *maps onto* $R_1 \times_R \cdots \times_R R_d$. *Thus*

$$g(R_1 \times_R \cdots \times_R R_d) \leq \sum_{i=1}^{d} [\lambda(S_i) - \lambda(R_i)] - (d-1)\lambda(R).$$

Proof. By Proposition 4.15, we see that if the triples (S_1, R_1, R) and (S_2, R_2, R) satisfy (CH), then the triple $(S_1 \#_R S_2, R_1 \times R_2, R)$ satisfies (CH). Hence by induction on d, we see that the triple $((((S_1 \#_R S_2) \#_R S_3) \#_R \cdots) \#_R S_d, R_1 \times_R \ldots \times_R R_d, R)$ satisfies (CH). Now, by induction on d, we can see that $\lambda((((S_1 \#_R S_2) \#_R S_3) \#_R \cdots) \#_R S_d) = \sum_{i=1}^{d} \lambda(S_i) - 2(d-1)\lambda(R)$ and $\lambda(R_1 \times_R \cdots \times_R R_d) = \sum_{i=1}^{d} \lambda(R_i) - (d-1)\lambda(R)$.

Thus

$$\begin{aligned} g(R_1 \times_R \cdots \times_R R_d) &\leq \lambda((((S_1 \#_R S_2) \#_R S_3) \#_R \cdots) \#_R S_d) - \lambda(R_1 \times_R \cdots \times_R R_d) \\ &= (\textstyle\sum_{i=1}^{d} \lambda(S_i) - 2(d-1)\lambda(R)) - (\textstyle\sum_{i=1}^{d} \lambda(R_i) - (d-1)\lambda(R)) \\ &= \textstyle\sum_{i=1}^{d} [\lambda(S_i) - \lambda(R_i)] - (d-1)\lambda(R). \end{aligned}$$

□

Let $J_i = \mathfrak{m}_i$. Then $0 :_{S_i} \mathfrak{m}_i = \operatorname{soc}(S_i)$ is contained in every non-zero ideal of S_i. Thus we get the following as immediate corollaries of Proposition 4.15.

Corollary 4.17. *Let* $(S_1, \mathfrak{m}_1, \mathsf{k})$, $(S_2, \mathfrak{m}_2, \mathsf{k})$ *be Gorenstein Artin local rings,* $(R_1, \mathfrak{m}_{R_1}, \mathsf{k})$, $(R_2, \mathfrak{m}_{R_2}, \mathsf{k})$ *be Artinian local rings such that each* R_i *is a quotient of* S_i *by a non-zero ideal, say* I_i. *Then* $S_1 \#_{\mathsf{k}} S_2$ *maps onto* $R_1 \times_{\mathsf{k}} R_2$.

Proof. The fact that $I_i \neq 0$ and $\operatorname{soc}(S_i) \subseteq I_i$ ensures that the triples (S_i, R_i, k) satisfy (CH). Thus the corollary follows from Proposition 4.15. □

Corollary 4.18. *Let $(R'', \mathfrak{m}, \mathsf{k})$ be a Teter ring. If $(R', \mathfrak{m}', \mathsf{k})$ is any Artinian local ring and S' is a Gorenstein Artin local ring mapping onto R' such that $\lambda(R') < \lambda(S')$, then $g(R' \times_{\mathsf{k}} R'') \leq \lambda(S') - \lambda(R')$.*
In particular, if R' is not Gorenstein, then $g(R' \times_{\mathsf{k}} R'') \leq g(R')$.

Proof. Let S'' be a Gorenstein Artin local ring such that $R'' \simeq S''/\mathrm{soc}(S'')$. Since $\lambda(S'') - \lambda(R'') = 1$ and $\lambda(\mathsf{k}) = 1$, by Proposition 4.15, $g(R' \times_{\mathsf{k}} R'') \leq \lambda(S') - \lambda(R')$.

The last part can be proved by taking S' to be a Gorenstein Artin local ring mapping onto R' such that $\lambda(S') - \lambda(R') = g(R')$. □

4.3 Some Special Cases

In this section we look more closely at fibre products and connected sums of Artinian quotients of polynomial rings over a field k.

Theorem 4.19. *Let $R_1 = \mathsf{k}[x_1, \ldots, x_m]/J_1$ and $R_2 = \mathsf{k}[y_1, \ldots, y_n]/J_2$ be Artinian local quotients of polynomial rings over k. Then*

$$R_1 \times_{\mathsf{k}} R_2 \simeq \mathsf{k}[\underline{x}, \underline{y}]/(J_1, J_2, x_i y_j : 1 \leq i \leq m, 1 \leq j \leq n).$$

Proof. Let $T = \mathsf{k}[\underline{x}, \underline{y}]$. Note that $R_1 \simeq T/(J_1, \underline{y})$, $R_2 \simeq T/(J_2, \underline{x})$, $(J_1, \underline{y}) + (J_2, \underline{x}) = (\underline{x}, \underline{y})$ and $(J_1, \underline{y}) \cap (J_2, \underline{x}) = (J_1, J_2, x_i y_j)$. Hence the theorem follows by Corollary 4.4 by taking $R' = T/(J_1, J_2, x_i y_j)$, $I_1 = (J_1, y_1, \ldots, y_n)$ and $I_2 = (J_2, x_1, \ldots, x_m)$. □

The following corollary is immediate from the above theorem.

Corollary 4.20. *Let R_1 and R_2 be standard graded Artinian local quotients of polynomial rings over k. Then $R_1 \times_{\mathsf{k}} R_2$ is also graded.*

Moreover, if the Hilbert series of R_1 and R_2 are $1+\sum_{i=1}^{m} h_i t^i$ and $1+\sum_{i=1}^{n} h_i' t^i$ respectively, then the Hilbert function of $R_1 \times_{\mathsf{k}} R_2$ is $1+\sum_{i=1}^{m} h_i t^i + \sum_{i=1}^{n} h_i' t^i$. Thus

$$H(R_1 \times_{\mathsf{k}} R_2, t) = H(R_1,t) + H(R_2,t) - 1 = H(R_1,t) + H(R_2,t) - H(\mathsf{k},t).$$

Remark 4.21. Let $(S_1, \mathfrak{m}_1, \mathsf{k})$ and $(S_2, \mathfrak{m}_2, \mathsf{k})$ be two Gorenstein Artin local rings that are not isomorphic to k. By Example 4.6(1), the pairs (S_1, k) and (S_2, k) satisfy (CH) and hence we can define $S_1 \#_{\mathsf{k}} S_2$.

Let $(\Delta_i) = \operatorname{soc}(S_i) = (0 :_{S_i} \mathfrak{m}_i)$. Then $S_1 \#_{\mathsf{k}} S_2 = (S_1 \times_{\mathsf{k}} S_2)/(\Delta_1, -\Delta_2)(S_1 \times_{\mathsf{k}} S_2)$.

Theorem 4.22. *Let $S_1 = \mathsf{k}[x_1, \ldots, x_m]/J_1$ and $S_2 = \mathsf{k}[y_1, \ldots, y_n]/J_2$ be Gorenstein Artin local quotients of polynomial rings over k. Then*

$$S_1 \#_{\mathsf{k}} S_2 \simeq \mathsf{k}[\underline{x}, \underline{y}]/(J_1, J_2, \Delta_1 - \Delta_2, x_i y_j : 1 \le i \le m, 1 \le j \le n),$$

where $(\Delta_i) = \operatorname{soc}(S_i)$.

Proof. By the previous remark, we see that $S_1 \#_{\mathsf{k}} S_2 = (S_1 \times_{\mathsf{k}} S_2)/(\Delta_1, -\Delta_2)(S_1 \times_{\mathsf{k}} S_2)$. By Theorem 4.19, we know that $S_1 \times_{\mathsf{k}} S_2 \simeq \mathsf{k}[\underline{x}, \underline{y}]/(J_1, J_2, x_i y_j : 1 \le i \le m, 1 \le j \le n)$, which proves the theorem. $\square$

As an immediate corollary, we see that the following holds.

Corollary 4.23. *Let S_1 and S_2 be standard graded Artinian local quotients of polynomial rings over k such that $\operatorname{Max}(S_1) = \operatorname{Max}(S_2)$. Then $S_1 \#_{\mathsf{k}} S_2$ is also graded.*

Furthermore, if the Hilbert series of S_1 and S_2 are $1+\sum_{i=1}^{m} h_i t^i$ and $1+\sum_{i=1}^{m} h_i' t^i$ respectively where $m = \operatorname{Max}(S_1 = \operatorname{Max}(S_2)$, then the Hilbert series of $S_1 \#_{\mathsf{k}} S_2$ is $1+\sum_{i=1}^{m} h_i t^i + \sum_{i=1}^{m} h_i' t^i - t^m$. Thus

$$H(S_1 \#_{\mathsf{k}} S_2, t) = H(S_1,t) + H(S_2,t) - 1 - t^m = H(S_1,t) + H(S_2,t) - (1-t^m) H(\mathsf{k},t).$$

Remark 4.24. Let k be a field of characteristic zero, S_1 and S_2 be Gorenstein Artin local quotients of $\mathsf{k}[x_1,\ldots,x_m]$ and $\mathsf{k}[y_1,\ldots,y_n]$ respectively. Let $E_1 = \mathsf{k}[X_1,\ldots,X_m]$ and $E_2 = \mathsf{k}[Y_1,\ldots,Y_n]$. With the notation as in Section 1.4, there are polynomials $F(X_1,\ldots,X_m) \in E_1$ and $G(Y_1,\ldots,Y_n) \in E_2$ which correspond to S_1 and S_2 respectively.

Since $S_1 \#_{\mathsf{k}} S_2$ is a Gorenstein Artin quotient of $\mathsf{k}[\underline{x},\underline{y}]$, it corresponds to a polynomial $H(\underline{X},\underline{Y}) \in E = \mathsf{k}[\underline{X},\underline{Y}]$.

Using Theorem 4.22, it can be easily seen that $H = F + G$.

4.4 Further Applications of Connected Sums

In Section 4.2, we used connected sums to give bounds on the Gorenstein colength of fibre products. In this section, we study some more applications of connected sums.

The First Application

The first application of connected sums is to prove the following:

Theorem 4.25. *Let $(R,\mathfrak{m},\mathsf{k})$ be an Artinian local ring which is not Gorenstein. Let $n \geq 0$ be any integer. Then there is a Gorenstein Artin ring S mapping onto R such that $\lambda(S) = n + g(R) + \lambda(R)$.*

Proof. There is a Gorenstein Artin ring $S' \neq R$ mapping onto R such that $\lambda(S') - \lambda(R) = g(R)$, by definition of $g(R)$. Since $S' \neq R$, we can write $R = S'/I$ for some non-zero ideal I in S'. Since every non-zero ideal in S' contains $\operatorname{soc}(S')$, $S'/\operatorname{soc}(S')$ maps onto R.

Let $S'' = \mathsf{k}[x]/(x^{n+2})$ where x is an indeterminate over k. Then $\lambda(S'') = n+2$. By Remark 4.13(2), $S = S' \#_{\mathsf{k}} S''$ maps onto $S'/\operatorname{soc}(S')$ and hence onto R. Now $\lambda(S) = \lambda(S'') + \lambda(S') - 2 = n + g(R) + \lambda(R)$, proving the theorem. $\square$

From the proof of the above theorem one can extract the following remark by taking $n=1$.

Remark 4.26. Let $(R,\mathfrak{m},\mathsf{k})$ be an Artinian local ring and $S \neq R$ be a Gorenstein Artin ring mapping onto R. Then there is a Gorenstein Artin local ring S'' mapping onto R such that $\lambda(S'') = \lambda(S)+1$.

The Second Application

In the following theorem, we prove the $n=2$ case of Theorem 3.16 for an arbitrary regular local ring. We prove:

Theorem 4.27. *Let $(T,\mathfrak{m},\mathsf{k})$ be a regular sequence of dimension d and $f_1,\ldots,f_d$ be a system of parameters in T. Then $g(T/(f_1,\ldots,f_d)^2) \leq \lambda(T/(f_1,\ldots,f_d))$.*

Before we begin the proof, we would like to state the key properties being used in this proof.

Remark 4.28. With notations as in Theorem 4.27, let $\mathfrak{b}_i = (f_1,\ldots,f_i)^2 + (f_{i+1},\ldots,f_d)$ and $\mathfrak{c}_i = (f_1,\ldots,f_{i-1},f_i^2,f_{i+1},\ldots,f_d)$.

1. For each i, $\mathfrak{b}_i \cap \mathfrak{c}_{i+1} = \mathfrak{b}_{i+1}$ and $\mathfrak{b}_i + \mathfrak{c}_{i+1} = (f_1,\ldots,f_d)$.
2. Let $S_i = T/(f_1,\ldots,f_{i-1},f_i^3,f_{i+1},\ldots,f_d)$. Then S_i is a Gorenstein Artin ring mapping onto $T/\mathfrak{c}_i$. Furthermore $\lambda(S_i) - \lambda(T/\mathfrak{c}_i)$
$= \lambda((f_1,\ldots,f_{i-1},f_i^2,f_{i+1},\ldots,f_d)/(f_1,\ldots,f_{i-1},f_i^3,f_{i+1},\ldots,f_d)) = \lambda(T/(f_1,\ldots,f_d))$.

Proof of Theorem 4.27. Let $R = T/(f_1,\ldots,f_d)$. With notation as in the above remark, we see by Corollary 4.4 and Remark 4.28(1) that for each i, $T/\mathfrak{b}_i \simeq T/\mathfrak{b}_{i-1} \times_R T/\mathfrak{c}_i$. Thus, we have

$$T/(f_1,\ldots,f_d)^2 = T/\mathfrak{b}_d \simeq (((T/\mathfrak{c}_1 \times_R T/\mathfrak{c}_2) \times_R T/\mathfrak{c}_3) \times_R \cdots) \times_R T/\mathfrak{c}_d.$$

Since $(S_i, T/\mathfrak{c}_i, R)$, $i = 1,\dots,d$, satisfy (CH), Corollary 4.16 shows that $g(T/(f_1,\dots,f_d)^2) \leq \sum_{i=1}^{d}[\lambda(S_i) - \lambda(T/\mathfrak{c}_i)] - (d-1)\lambda(R)$. Thus, by Remark 4.28(2), $g(T/(f_1,\dots,f_d)^2) \leq \lambda(R)$, proving the theorem. □

The Third Application

The main goal in the rest of this section is to prove the following extension of the Huneke-Vraciu theorem (Theorem 0.3):

Theorem 4.29. *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring with canonical module ω. Assume further that 2 is invertible in R. If ω maps onto $\mathfrak{m}$, then $g(R) \leq 1$.*

We use connected sums and the following remark (which is a corollary of Teter's theorem) in the proof of Theorem 4.29.

Remark 4.30. Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring of length 2. Then

1) R is Gorenstein.

2) R is a Teter ring.

Lemma 4.31. *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring and $\mathfrak{a}$, $\mathfrak{a}'$ and $\mathfrak{a}''$ ideals in R such that $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$. Then $\omega_R = (0 :_{\omega_R} \mathfrak{a}') + (0 :_{\omega_R} \mathfrak{a}'')$.*

Proof. Since $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$, $(0 :_{\omega_R} \mathfrak{a}) = (0 :_{\omega_R} \mathfrak{a}') \cap (0 :_{\omega_R} \mathfrak{a}''')$. Thus the fact that $(0 :_{\omega_R} \mathfrak{a}) \simeq \omega_{R/\mathfrak{a}}$ and

$$\lambda((0 :_{\omega_R} \mathfrak{a}') \cap (0 :_{\omega_R} \mathfrak{a}'')) + \lambda((0 :_{\omega_R} \mathfrak{a}') + (0 :_{\omega_R} \mathfrak{a}'')) = \lambda((0 :_{\omega_R} \mathfrak{a}')) + \lambda((0 :_{\omega_R} \mathfrak{a}''))$$

give us

$$\begin{aligned}
\lambda(\omega_{R/\mathfrak{a}})+\lambda((0:_{\omega_R}\mathfrak{a}')+(0:_{\omega_R}\mathfrak{a}'')) &= \lambda(R/\mathfrak{a}')+\lambda(R/\mathfrak{a}'') \\
&= 2\lambda(R)-(\lambda(\mathfrak{a}')+\lambda(\mathfrak{a}'')) \\
&= 2\lambda(R)-(\lambda(\mathfrak{a}'+\mathfrak{a}'')+\lambda(\mathfrak{a}'\cap\mathfrak{a}'')) \\
&= \lambda(R)+\lambda(R/\mathfrak{a}),
\end{aligned}$$

since $\mathfrak{a}'+\mathfrak{a}''=\mathfrak{a}$ and $\mathfrak{a}'\cap\mathfrak{a}''=0$. This proves the lemma since $\lambda(R/\mathfrak{a})=\lambda(\omega_{R/\mathfrak{a}})$ and $\lambda(R)=\lambda(\omega_R)$. □

Proposition 4.32. *Let $(R,\mathfrak{m},\mathsf{k})$ be an Artinian local ring, $I=\langle\operatorname{soc}(R)\cap(\mathfrak{m}\setminus\mathfrak{m}^2)\rangle$. If ω_R maps onto $\mathfrak{m}$, then $\omega_{R/I}$ maps onto $\mathfrak{m}/I$.*

Proof. Let $\mathfrak{m}$ be minimally generated by $(x_1,\ldots,x_r,y_1,\ldots,y_s)$, where $x_i\notin\operatorname{soc}(R)$ and $y_j\in\operatorname{soc}(R)$. Then $I=(y_1,\ldots,y_s)\simeq\mathsf{k}^s$. Let $I'=(x_1,\ldots,x_r)$. Note that $\mathfrak{m}=I\oplus I'$. Then $\omega_{R/I}\simeq(0:_{\omega_R}I)$ and $\omega_{R/I'}\simeq(0:_{\omega_R}I')$. Thus, to prove the proposition, it is enough to show that $(0:_{\omega_R}I)$ maps onto $\mathfrak{m}/I$.

<u>Claim 1:</u> $\phi((0:_{\omega_R}I'))\subseteq\operatorname{soc}(R)$.

Since $\mathfrak{m}^2(R/I')=0$, $\mathfrak{m}^2(0:_{\omega_R}I')=0$. Thus $\mathfrak{m}(0:_{\omega_R}I')\in\operatorname{soc}(\omega)$. Note that by counting lengths, $\ker(\phi)\neq 0$, hence $\operatorname{soc}(\omega)\subseteq\ker(\phi)$. Thus $\phi(\mathfrak{m}(0:_{\omega_R}I'))=0$, i.e., $\phi((0:_{\omega_R}I'))\subseteq\operatorname{soc}(R)$.

Let $\pi:\mathfrak{m}\longrightarrow\mathfrak{m}/I$ be the natural projection and let α denote the surjective map $\pi\circ\phi:\omega_R\longrightarrow\mathfrak{m}/I$.

<u>Claim 2:</u> $\alpha(0:_{\omega_R}I)=\mathfrak{m}/I$.

As $\mathfrak{m}=I\oplus I'$, by Lemma 4.31, we have $\omega_R=(0:_{\omega_R}I)+(0:_{\omega_R}I')$. Hence $\alpha(0:_{\omega_R}I)+\alpha(0:_{\omega_R}I')=\mathfrak{m}/I$. But by Claim (1), $\alpha(0:_{\omega_R}I')\subseteq\operatorname{soc}(R/I)$. Since $\operatorname{soc}(R/I)\subseteq\mathfrak{m}(\mathfrak{m}/I)$, we have $\mathfrak{m}/I=\alpha(0:_{\omega_R}I)+\mathfrak{m}(\mathfrak{m}/I)$. Therefore, by NAK, Claim (2) is proved. Thus $\omega_{R/I}\simeq(0:_{\omega_R}I)$ maps onto $\mathfrak{m}/I$. □

The following is a well-known theorem. We give a different proof using connected sums.

Theorem 4.33. *Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring such that $\mathfrak{m}^2 = 0$. Then R is a Teter ring. Moreover, if $\mu(\mathfrak{m}) \geq 2$, then $g(R) = 1$.*

Proof. Let $\mathfrak{m}$ be minimally generated by $(y_1, \ldots, y_s)$. Induce on s. If $s = 1$, then by Remark 4.30, R is a Teter ring.

Suppose $s \geq 2$. Set $R' = R/(y_1, \ldots, y_{s-1})$ and $R'' = R/(y_s)$. Then Corollary 4.4, $R \simeq R' \times_{\mathsf{k}} R''$. Since $\lambda(R') = 2$, R' is a Teter ring. By induction, R'' is a Teter ring. Write $R' = S'/\mathrm{soc}(S')$ and $R'' \simeq S''/\mathrm{soc}(S'')$, where S' and S'' are Gorenstein Artin rings. Therefore, by Corollary 4.17, $S' \#_{\mathsf{k}} S''$ is a Gorenstein Artin local ring mapping onto $R \simeq R' \times_{\mathsf{k}} R''$. Moreover, since $\lambda(S' \#_{\mathsf{k}} S'') - \lambda(R) = 1$, we see that R is a Teter ring.

If $\mu(\mathfrak{m}) = s \geq 2$, then R is not Gorenstein. Hence $g(R) = 1$ in this case. $\square$

We are now ready to prove Theorem 4.29.

Proof of Theorem 4.29. Let $\mathfrak{m}$ be minimally generated by $(x_1, \ldots, x_r, y_1, \ldots, y_s)$, where $x_i \notin \mathrm{soc}(R)$ and $y_j \in \mathrm{soc}(R)$. Let $I = (y_1, \ldots, y_s)$ and $I' = (x_1, \ldots, x_r)$. Since $\mathfrak{m} = I \oplus I'$, by Corollary 4.4, $R \simeq R/I \times_{\mathsf{k}} R/I'$.

If $r = 0$, then the result follows from Theorem 4.33. If s = 0, then the theorem follows from the Huneke-Vraciu theorem. Hence we can assume that both I and I' are non-zero ideals in R.

Now, by Proposition 4.32, $\omega_{R/I}$ maps onto $\mathfrak{m}/I$. Moreover $\mathrm{soc}(R/I) \subseteq (\mathfrak{m}/I)^2$. Hence by the Huneke-Vraciu theorem, there is a Gorenstein Artin ring S mapping onto R/I such that $\lambda(S) - \lambda(R/I) = 1$. Moreover, by Theorem 4.33, R/I' is a Teter ring. Hence by Corollary 4.18, $g(R) \leq \lambda(S) - \lambda(R/I) = 1$. $\square$

Chapter 5

Second Foundation

5.1 Introduction

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with infinite residue field k. Let I be an $\mathfrak{m}$-primary ideal and $J = (x_1, \dots, x_d)$ be a minimal reduction of I. In [23], P. Valabrega and G. Valla show that the condition $I^n \cap J = JI^{n-1}$ holds for all n if and only if the associated graded ring $\mathrm{gr}_R(I) = R/I \oplus I/I^2 \oplus \cdots$ is Cohen-Macaulay.

In [20], M. Rossi studies the condition $J \cap I^k = JI^{k-1}$ for all $k \leq n$. We use the same terminology as Rossi, in particular the we define the following:

Definition 5.1. *([20])*

a) *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring, I an $\mathfrak{m}$-primary ideal and J be a minimal reduction of I. We say that I is n-standard with respect to J if $J \cap I^k = JI^{k-1}$ for all $k \leq n$.*

b) *We say is I is n-standard if I is n-standard with respect to every minimal reduction J of I.*

Remark 5.2.

1) Every ideal is 1-standard.

2) Let $(R,\mathfrak{m},\mathsf{k})$ be a Cohen-Macaulay local ring with infinite residue field, J be a minimal reduction of $\mathfrak{m}$. It is well-known that $\mathfrak{m}^2 \cap J = J\mathfrak{m}$, for example see Proposition 8.3.3(1) in [21]. Thus $\mathfrak{m}$ is 2-standard.

3) An n-standard ideal is k-standard for $1 \leq k \leq n$.

Remark 5.3.

1. Let $(R,\mathfrak{m},\mathsf{k})$ be a Cohen-Macaulay local ring, J a minimal reduction of $\mathfrak{m}$ and $\widehat{R}$ be the $\mathfrak{m}$-adic completion. Since the extension $R \longrightarrow \widehat{R}$ is faithfully flat, $J \cap \mathfrak{m}^n = J\mathfrak{m}^{m-1}$ if and only if $\widehat{J} \cap \widehat{\mathfrak{m}}^n = \widehat{J}\,\widehat{\mathfrak{m}}^{n-1}$, where $\widehat{\_} = \_ \otimes \widehat{R}$.

2. By Lemma 8.1.3 in [21], we see that J is a reduction of $\mathfrak{m}$ if and only if $\widehat{J}$ is a reduction of $\widehat{\mathfrak{m}}$. Hence by (1), $\mathfrak{m}$ is n-standard with respect to J in R if and only if $\widehat{\mathfrak{m}}$ is n-standard with respect to $\widehat{J}$ in $\widehat{R}$.

5.2 Preliminaries

Koszul Homology

Let $G = \oplus_{i \geq 0} G_i$ be a graded ring with $x_1, \ldots, x_d \in G_1$. Let $\mathbf{K}_\bullet(x_1,\ldots,x_k;G)$ be the Koszul complex on $x_1,\ldots,x_k$ over G. Then $\mathbf{K}_\bullet(x_1,\ldots,x_k;G)$ is:

$$0 \to G[-k] \to G[-k+1]^{\oplus d} \to \cdots \to G[-2]^{\oplus \binom{d}{2}} \to G[-1]^{\oplus d} \xrightarrow{(x_1,\ldots,x_k)} G \to 0$$

Remark 5.4.

1. There is a short exact sequence of complexes

$$0 \longrightarrow \mathbf{K}_\bullet(x_1,\ldots,x_{k-1};G) \longrightarrow \mathbf{K}_\bullet(x_1,\ldots,x_k;G) \longrightarrow \mathbf{K}_\bullet(x_1,\ldots,x_{k-1};G)[-1] \longrightarrow 0.$$

2. Let $H_i(\_;G)$ be the ith Koszul homolgy. The above short exact sequence of Koszul complexes gives a long exact sequence on the Koszul homologies:

$$H_i(x_1,\dots,x_{k-1};G) \xrightarrow{\cdot x_k} H_i(x_1,\dots,x_{k-1};G) \to H_i(x_1,\dots,x_k;G) \to H_{i-1}(x_1,\dots,x_{k-1};G) \xrightarrow{\cdot x_k}$$

3. Let $H_i(x_1,\dots,x_k)_j$ be the jth graded piece of H_i. Since all the maps in the above long exact sequence are of degree zero and $\deg(x_k)=1$, the long exact sequence breaks up into the following graded pieces:

$$H_i(x_1,\dots,x_{k-1})_{j-1} \xrightarrow{\cdot x_k} H_i(x_1,\dots,x_{k-1})_j \to H_i(x_1,\dots,x_k)_j \to H_{i-1}(x_1,\dots,x_{k-1})_{j-1} \xrightarrow{\cdot x_k}$$

4. Notice that the ith Koszul homology $H_i(x_1,\dots,x_k;G)$ is a subquotient of $G[-i]^{\oplus\binom{k}{i}}$, thus if the image of $(r_1,\dots,r_{\binom{k}{i}})$ is in $H_i(x_1,\dots,x_k;G)_j$, then $\deg(r_l)=j-i$ as an element of G.

5. We have, $H_i(x_1,\dots,x_k;G)=0$ for $i>k$. By (4), we also see that $H_i(\_;G)_j=0$ for $j<i$.

6. The vector $\overline{(r_1,\dots,r_k)}$ is zero in $H_1(x_1,\dots,x_k;G)$ if it can be written as a linear combination of the Koszul relations, i.e., as elements in $G^{\oplus k}$,

$$\begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_j \\ \vdots \\ r_k \end{pmatrix} = \sum_{1 \le i < j \le k} s_{ij} \begin{pmatrix} 0 \\ \vdots \\ x_j \\ \vdots \\ -x_i \\ \vdots \\ 0 \end{pmatrix} \text{ where } s_{ij} \in G.$$

Rearranging, we see that this happens if and only if $(r_1, \ldots, r_k) = (x_1, \ldots, x_k)S$, where S is the skew-symmetric matrix

$$\begin{pmatrix} 0 & -s_{12} & \cdots & -s_{1k} \\ s_{12} & 0 & \cdots & -s_{2k} \\ \vdots & & \ddots & \vdots \\ s_{1k} & s_{2k} & \cdots & 0 \end{pmatrix}$$

Lemma 5.5. *With notations as in the above remark, if $H_i(x_1, \ldots, x_k; G)_j = 0$ for all $k \le n$, then $H_{i+1}(x_1, \ldots, x_k; G)_{j+1} = 0$ for all $k \le n$.*

Proof. By looking at the long exact sequence of the Koszul homologies, we see that $H_{i+1}(x_1, \ldots, x_k; G)_{j+1} = 0$ if $H_i(x_1, \ldots, x_k; G)_j = 0$ and $H_{i+1}(x_1, \ldots, x_{k-1}; G)_{j+1} = 0$. Now $H_i(x_1, \ldots, x_k; G)_j = 0$ by the hypothesis and $H_{i+1}(x_1, \ldots, x_{k-1}; G)_{j+1} = 0$ by induction on k. □

We use the following notion in Proposition 5.8.

Definition 5.6. *We say that a ring R is connected in codimension 1 if given any two minimal primes $\mathfrak{p}$ and $\mathfrak{q}$ in R, there is a sequence of minimal primes $\mathfrak{p} = \mathfrak{p}_1, \ldots, \mathfrak{p}_k = \mathfrak{q}$ such that* $\mathrm{ht}(\mathfrak{p}_i + \mathfrak{p}_{i+1}) \le 1$.

Remark 5.7. We use this remark in our proof of Propostion 5.8.

1. Let $(x_1,\ldots,x_d)$ be a system of parameters in a Noetherian ring G. Let $y_i = x_i + \sum_{j=r+1}^{d} a_{ij}x_j$ for $i = 1,\ldots,r$ and $y_i = x_i$ for $r < i \le d$ where $a_{ij} \in G$. Then, since $(y_1,\ldots,y_d) = (x_1,\ldots,x_d)$, $(y_1,\ldots,y_d)$ is also a system of parameters. Thus we see that $\operatorname{ht}(y_1,\ldots,y_r) = r$.

2. Let S be a matrix of units in G such that $\operatorname{rank}(S) = r$. Let I be the ideal generated by the d components of the vector $(x_1,\ldots,x_d)S$. Without loss of generality, we may assume that S is in its reduced row echelon form. Observe that by reordering the x's if necessary, $I = (y_1,\ldots,y_r)$, where y_i is of the form $x_i + \sum_{j=r+1}^{d} a_{ij}x_j$ for $i = 1,\ldots,r$. Thus, by (1), $\operatorname{ht}(I) = r = \operatorname{rank}(S)$.

Proposition 5.8. *Let $G = \oplus_{i\ge 0} G_i$ be a graded ring with $x_1,\ldots,x_d \in G_1$. Suppose that G is reduced and connected in codimension 1. Let $\operatorname{Min}(G) = \{\mathfrak{p}_1,\ldots,\mathfrak{p}_l\}$ be the set of minimal primes of G. If $H_1(x_1,\ldots,x_k;G/\mathfrak{p}_i)_{\le 2} = 0$ for $i = 1,\ldots,l$, then $H_1(x_1,\ldots,x_k;G)_{\le 2} = 0$.*

Proof. Let $\overline{(r_1,\ldots,r_k)} \in H_1(x_1,\ldots,x_k;G)_{\le 2}$, i.e., $\sum_{i=1}^{k} r_i x_i = 0$ in G with $\deg(r_i) < 2$. Let $^-$ denote going modulo $\mathfrak{p}_i$. Since $\overline{(\bar{r}_1,\ldots,\bar{r}_k)} \in H_1(x_1,\ldots,x_k;G/\mathfrak{p}_i)_{\le 2} = 0$ for each i, we can write $(r_1,\ldots,r_k) = (x_1,\ldots,x_k)S_i + (p_{i1},\ldots,p_{ik})$, where S_i is a skew-symmetric matrix with entries in G and $p_{in} \in \mathfrak{p}_i$, $n = 1,\ldots,k$.

Now since G is connected in codimension 1, given any two minimal primes $\mathfrak{p}$ and $\mathfrak{q}$ in G, there is a sequence of ideals $\mathfrak{p} = \mathfrak{p}_{i1},\ldots,\mathfrak{p}_{im} = \mathfrak{q}$ such that $\operatorname{ht}(\mathfrak{p}_{in} + \mathfrak{p}_{in+1}) \le 1$.

<u>Claim:</u> $S_{in} = S_{in+1}$.

If $S_{in} \ne S_{in+1}$, then $S_{in} - S_{in+1}$ is a non-zero skew-symmetric matrix, i.e., it has a 2×2 minor of the form $\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}$ where $r \ne 0$. Thus $\operatorname{rank}(S_{in} - S_{in+1}) \ge 2$. Since $(x_1,\ldots,x_k)(S_{in} - S_{in+1}) \in \mathfrak{p}_{in} + \mathfrak{p}_{i,n+1}$, we see by Remark 5.7(2) that $\operatorname{rank}(S_{in} - S_{in+1}) \ge 2$ forces $\operatorname{ht}(\mathfrak{p}_{in} + \mathfrak{p}_{in+1}) \ge 2$. This contradiction proves the claim.

Thus we have $S_{\mathfrak{p}} = S_{\mathfrak{q}}$. Since $(p_1,\ldots,p_k)-(q_1,\ldots,q_k) = (x_1,\ldots,x_k)(S_{\mathfrak{p}}-S_{\mathfrak{q}})$, this forces $(p_{m1},\ldots,p_{mk}) = (p_{n1},\ldots,p_{nk})$ for $1 \le m,n \le l$.

Let $S_i = S$ and $(p_{i1},\ldots,p_{ik}) = (p_1,\ldots,p_k)$, $i = 1,\ldots,l$. Thus $p_n \in \cap_{i=1}^{l}\mathfrak{p}_i = 0$ for $n = 1,\ldots,k$, since G is reduced. Therefore we have $(r_1,\ldots,r_k) = (x_1,\ldots,x_k)S$. This proves that $\overline{(r_1,\ldots,r_k)} = 0$ in $H_1(x_1,\ldots,x_k;G)$ proving the lemma. $\square$

Proposition 5.9. *Let $G = \oplus_{i\ge 0}G_i$ be a graded ring with $x_1,\ldots,x_d \in G_1$. Then with notations as above, $H_1(x_1,\ldots,x_k;G)_{\le n} = 0$ for $1 \le k \le l$ if and only if $(x_1,\ldots,x_{k-1}) : x_k \subseteq (x_1,\ldots,x_{k-1}) + \oplus_{i\ge n}G_i$ for $1 \le k \le l$.*

Proof. Firstly assume that $(x_1,\ldots,x_{k-1}) : x_k \subseteq (x_1,\ldots,x_{k-1}) + \oplus_{i\ge n}G_i$ for $1 \le k \le l$. We want to prove that $H_1(x_1,\ldots,x_k;G)_{\le n} = 0$ by induction on k.

When $k = 1$, we see that $(0 :_G x_1) \subseteq \oplus_{i\ge n}G_i$. Note that $H_1(x_1;G) \simeq (0 :_G x_1)[-1]$. Hence $H_1(x_1;G)_{\le n} = 0$.

Let $\overline{(r_1,\ldots,r_k)} \in H_1(x_1,\ldots,x_k;G)_j$ for some $j \le n$, where $k > 1$ Thus we have $r_i \in G_{j-1}$ and $\sum_{i=1}^{k} r_i x_i = 0$ in G_j. Thus $r_k \in (x_1,\ldots,x_{k-1}) :_G x_k \subseteq (x_1,\ldots,x_{k-1}) + \oplus_{i\ge n}G_i$ by assumption. Thus $r_k = \sum_{i=1}^{k-1} s_i x_i + s_k$ where $s_k \in \oplus_{i\ge n}G_i$. By degree arguments, we may assume that $s_i \in G_{j-2}$, $i = 1,\ldots,k-1$ and $s_k = 0$ in G.

Thus $0 = \sum_{i=1}^{k-1}(r_i + s_i x_k)x_i$. By induction $\overline{(r_1,\ldots,r_{k-1})} + \overline{(x_k s_1,\ldots,x_k s_{k-1})} = 0$ in $H_1(x_1,\ldots,x_{k-1};G)$. Thus, by Remark 5.4(6), we see that there is a $(k-1)\times(k-1)$ skew-symmetric matrix S with entries in G such that $(r_1,\ldots,r_{k-1}) + (x_k s_1,\ldots,x_k s_{k-1}) = (x_1,\ldots,x_{k-1})S$. We also know that $r_k = \sum_{i=1}^{k-1} s_i x_i$. Hence we have

$$\begin{pmatrix} r_1 \\ \vdots \\ \vdots \\ r_k \end{pmatrix} = \left(\begin{array}{ccc|c} & & & -s_1 \\ & S & & -s_2 \\ & & & \\ \hline s_1 & s_2 & \cdots & 0 \end{array}\right) \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_k \end{pmatrix}.$$

which shows by Remark 5.4(6) that $\overline{(r_1,\dots,r_k)} = 0$ in $H_1(x_1,\dots,x_k;G)$. □

The Associated Graded Ring

Let $G = \mathrm{gr}_R(I) = R/I \oplus I/I^2 \oplus \cdots$ be the associated graded ring of I. If $s \in R$ is an element such that $s \in I^k \setminus I^{k+1}$, we let s' denote $s + I^{k+1}$, the leading form of s in G. Let $J = (x_1,\dots,x_d)$ be a minimal reduction of I.

Proposition 5.10. *Let $(R,\mathfrak{m},\mathsf{k})$ be a Cohen-Macaulay local ring, I an $\mathfrak{m}$-primary ideal and $J = (x_1,\dots,x_d)$ a minimal reduction of I and $G = \mathrm{gr}_R(I)$ be the graded ring associated to I. With notations as in the discussion above, the ideal I is n-standard with respect to J if and only if $H_1(x_1',\dots,x_k';G)_j = 0$ for $j < n$ and $0 \le k \le d$.*

Proof. Assume that I is n-standard with respect to J. Suppose that for $j < n$ and $0 \le k \le d$, $\overline{(r_1',\dots,r_k')} \in H_1(x_1',\dots,x_k';G)_j$, i.e., $\sum_{i=1}^{l} r_i'x_i' = 0$ in G, where $\deg(r_i') = j-1$. Thus $\sum r_ix_i \in I^{j+1} \cap J = JI^j$, i.e., we can write $\sum_{i=1}^{k} r_ix_i = \sum_{i=1}^{d} s_ix_i$, where $s_i \in I^j$. Thus there is a skew-symmetric $k \times k$ matrix S_k such that

$$(r_1,\dots,r_k) = (x_1,\dots,x_k)S_k + (s_1,\dots,s_k).$$

Thus $(r_1',\dots,r_k') = (x_1',\dots,x_k')S_k'$ in $G_{j-1}^{\oplus d}$, which means that $\overline{(r_1',\dots,r_k')} = 0$ in $H_1(x_1',\dots,x_k';G)_j$ for $j < n$.

Conversely, suppose $\sum_{i=1}^{k} r_ix_i \in I^j$ for some $j \le n$, with $r_i \notin I^{j-1}$. Then $\sum_{i=1}^{k} r_i'x_i' = 0$ in $G_{\le j-1}$. Thus $\overline{(r_1',\dots,r_k')} \in H_1(x_1',\dots,x_k';G)_{\le j-1} = 0$, i.e., there is a skew-symmetric $k \times k$ matrix S_k with entries in R such that $(r_1',\dots,r_k') = (x_1',\dots,x_k')S_k'$ in $G_{\le j-1}^{\oplus d}$, i.e., $(r_1,\dots,r_k) = (x_1,\dots,x_k)S_k + (s_1,\dots,s_k)$ for some $s_i \in I^{j-1}$. Thus $\sum r_ix_i = \sum s_ix_i \in JI^{j-1}$, for each $j \le n$, i.e., I is n-standard with respect to J. □

As a consequence, we get the following theorem of Valabrega and Valla([23], Theorem 2.3).

Corollary 5.11. *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a Cohen-Macaulay local ring, I an* $\mathfrak{m}$*-primary ideal and* $J = (x_1, \ldots, x_d)$ *a minimal reduction of I and* $G = \mathrm{gr}_R(I)$ *be the graded ring associated to I. With notations as in the discussion above, then* $x_1', \ldots, x_d'$ *is a regular sequence in G (and hence G is Cohen-Macaulay) if and only if* $I^n \cap J = I^{n-1}J$ *for all n.*

Corollary 5.12. *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a Cohen-Macaulay local ring with infinite residue field and* $J = (x_1, \ldots, x_d)$ *be a minimal reduction of* $\mathfrak{m}$*. With notations as above,* $H_i(x_1', \ldots, x_k'; G)_i = 0$ *for all k.*

Proof. As noted before in Remark 5.2(2), $J \cap \mathfrak{m}^2 = J\mathfrak{m}$. Hence by the Proposition 5.10, $H_1(x_1', \ldots, x_k'; G)_1 = 0$. The corollary follows by repeated application of the Lemma 5.5. $\square$

Chapter 6

Consequences of n-standardness and 3-standardness of the maximal ideal

6.1 Invariance of a Length Associated to Minimal Reductions

A general question one can ask is: Let $(R, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay local ring with infinite residue field, I an $\mathfrak{m}$-primary ideal and J a minimal reduction of I. Is $\lambda(I^n/JI^{n-1})$ independent of the minimal reduction J chosen?

A more specific question we ask is the following:

Question 6.1. Given a minimal reduction J of I, when is it true that

$$\lambda(I^{n+1}/JI^n) = e_0(I) + \sum_{i=0}^{n} (-1)^{i+1} \binom{d-1}{i} \lambda(I^{n-i}/I^{n-i+1})?$$

Remark 6.2. If $(R, \mathfrak{m})$ is a 1-dimensional Cohen-Macaulay local ring with infinite residue field, by Theorem 6.18 in [24], we see that $\lambda(\mathfrak{m}^{n+1}/J\mathfrak{m}^n) = e_0(R) - \lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ Thus, the Question 6.1 has a positive answer in the 1-dimensional case for $I = \mathfrak{m}$ since $\binom{d-1}{i} = 0$ for $i > 0$.

In his paper ([18], Example 2), T. Puthenpurakal gives the following example which shows that the above formula doesn't hold in general.

Example 6.3. Let $R = \mathsf{k}[|x,y|]$, $I = (x^7, x^6y, x^2y^5, y^7)$ and $J = (x^7, y^7)$. In this case, $d = \dim(R) = 2$. One can use a computer algebra package (we use Macaulay 2) to see that $\lambda(I^3/I^2J) = 3$ whereas $e_0(I) + \lambda(I/I^2) - \lambda(I^2/I^3) = 1$. Thus the above formula does not hold even for $n = 2$ in dimension 2.
Note that in this case $I^2 \cap J = (y^{14}, x^2y^{12}, x^4y^{10}, x^6y^8, x^7y^7, x^8y^6, x^9y^5, x^{12}y^2, x^{13}y, x^{14}) \neq (y^{14}, x^2y^{12}, x^6y^8, x^7y^7, x^9y^5, x^{13}y, x^{14}) = IJ$. Thus I is not 2-standard with respect to J.

In Theorem 6.5, we prove that $\lambda(I^{n+1}/JI^n)$ is independent of the minimal reduction chosen when I is n-standard with respect to J for every minimal reduction J of I by proving the above formula.

Proposition 6.4. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay local ring, I an $\mathfrak{m}$-primary ideal and J be a minimal reduction of I. If I is n-standard with respect to J, then for $1 \leq k \leq n$,* $\lambda(JI^{k-1}/JI^k) = d\lambda(I^{k-1}/I^k) - \binom{d}{2}\lambda(I^{k-2}/I^{k-1}) + \cdots + (-1)^{k-1}\binom{d}{k}\lambda(R/I)$,

$$\textit{i.e., } \lambda(JI^{k-1}/JI^k) = \sum_{i=1}^{k} (-1)^{i-1} \binom{d}{i} \lambda(I^{k-i}/I^{k-i+1}) \quad \textit{for } 1 \leq k \leq n.$$

Proof. Let $J = (x_1, \ldots, x_d)$. Since R is Cohen-Macaulay, $x_1, \ldots, x_d$ is a regular system of parameters (for example, by Corollary 8.3.9 in [21]). Consider the Koszul complex on the x's, which by restriction, gives us the following complex:

$$R^{\oplus\binom{d}{k}} \to I^{\oplus\binom{d}{k-1}} \to \cdots \to (I^{k-1})^{\oplus d} \overset{(x_1,\ldots,x_d)}{\to} JI^{k-1} \to 0.$$

Tensor the above complex with R/I to get the complex:

$$0 \to (R/I)^{\oplus\binom{d}{k}} \to (I/I^2)^{\oplus\binom{d}{k-1}} \to \cdots \to (I^{k-1}/I^k)^{\oplus d} \overset{(x_1,\dots,x_d)}{\to} (JI^{k-1}/JI^k) \to 0. \qquad (*)$$

Observe that the exactness of this complex gives the formula and hence proves the proposition.

When $k = 1$, the complex $(*)$ is $0 \longrightarrow (R/I)^{\oplus d} \overset{(x_1,\dots,x_d)}{\longrightarrow} (J/JI) \longrightarrow 0$. This is clearly surjective. Injectivity follows as follows: If $\sum r_i x_i \in IJ$, writing $\sum r_i x_i = \sum s_i x_i$ for $s_i \in I$, we see that there is a skew-symmetric matrix S with entries in R such that $(r_1,\dots,r_d) = (x_1,\dots,x_d)S + (s_1,\dots,s_d)$. Since $(x_1,\dots,x_d) \subseteq I$, we get $r_i \in I$, proving injectivity.

Let $H_i(*)$ be the ith homology of the complex $(*)$. For $k > 1$, consider the restriction $(I^{k-2})^{\oplus\binom{d}{2}} \longrightarrow (I^{k-1})^{\oplus d} \overset{(x_1,\dots,x_d)}{\longrightarrow} JI^{k-1} \longrightarrow 0$ of the Koszul complex on $x_1,\dots,x_d$. The fact that $x_1,\dots,x_d$ is a regular sequence in R forces the above complex to be exact. Tensoring with R/I, we see that $(I^{k-2}/I^{k-1})^{\oplus\binom{d}{2}} \to (I^{k-1}/I^k)^{\oplus d} \to (JI^{k-1}/JI^k) \to 0$ is exact by left-exactness of the tensor product. Hence $H_0(*)$ and $H_1(*)$ are zero.

Now the complex

$$0 \to (R/I)^{\oplus\binom{d}{k}} \to (I/I^2)^{\oplus\binom{d}{k-1}} \to \cdots \to (I^{k-1}/I^k)^{\oplus d}$$

is part of the degree k piece of the Koszul complex $\mathbf{K}_\bullet(x'_1,\dots,x'_d;G)$, where $G = \mathrm{gr}_R(I)$ is the graded ring associated with I and x'_i is the image of x_i in G_1. Thus the homology $H_i(*)$ is given by $H_i(x'_1,\dots,x'_d;G)_k$ for $i > 1$.

Since I is n-standard with respect to J, by Proposition 5.10, $H_1(x'_1,\dots,x'_d;G)_j = 0$ for $j < n$. Hence $H_i(x'_1,\dots,x'_d;G)_{j+i-1} = 0$ for $j < n$ by Lemma 5.5. In particular, $H_i(x'_1,\dots,x'_d;G)_k = 0$ for each $i \geq 2$. This shows that $(*)$ is exact proving the proposition. $\square$

The following theorem shows that Question 6.1 has a positive answer when I is n-standard with respect to J.

Theorem 6.5. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay local ring with an infinite residue field k. If I is an $\mathfrak{m}$-primary ideal and J is a minimal reduction of I such that $J \cap I^i = JI^{i-1}$ for $1 \leq i \leq n$, then for $0 \leq k \leq n$, $\lambda(I^{k+1}/JI^k) = e_0(I) - \lambda(I^k/I^{k+1}) + (d-1)\lambda(I^{k-1}/I^{k-2}) + \cdots + (-1)^k \binom{d-1}{k-1}\lambda(I/I^2) + (-1)^{k+1}\binom{d-1}{k}\lambda(R/I)$,*

$$\text{i.e., } \lambda(I^{k+1}/JI^k) = e_0(I) + \sum_{i=0}^{k}(-1)^{i+1}\binom{d-1}{i}\lambda(I^{k-i}/I^{k-i+1}). \quad \text{for } 0 \leq k \leq n \quad (\sharp)$$

Proof. We prove this by induction on n. For $n = 0$, since $e_0(I) = \lambda(R/J)$, $(\sharp)$ holds.

Assume $n \geq 1$. By induction, $\lambda(I^{k+1}/JI^k) = e_0(I) + \sum_{i=0}^{k}(-1)^{i+1}\binom{d-1}{i}\lambda(I^{k-i}/I^{k-i+1})$ for $0 \leq k \leq n-1$. Hence we need to prove $(\sharp)$ only for $k = n$.

We have $\lambda(I^{n+1}/JI^n) = \lambda(I^n/JI^{n-1}) + \lambda(JI^{n-1}/JI^{n-2}) - \lambda(I^n/I^{n+1})$. By induction, $\lambda(I^n/JI^{n-1}) = e_0(I) + \sum_{i=0}^{n-1}(-1)^{i+1}\binom{d-1}{i}\lambda(I^{n-i-1}/I^{n-i})$ and by Proposition 6.4, $\lambda(JI^{n-1}/JI^n) = \sum_{i=1}^{n}(-1)^{i-1}\binom{d}{i}\lambda(I^{n-i}/I^{n-i+1})$ since I is n-standard with respect to J. Combining these by using Pascal's identity $\binom{d}{i} - \binom{d-1}{i-1} = \binom{d-1}{i}$, we get $(\sharp)$ for $k = n$. □

One can see from the following example that n-standardness is not necessary for a positive answer to Question 6.1.

Example 6.6. Let $R = \mathsf{k}[x,y,z]_{(x,y,z)}/(xz - y^3, z^2)$. Then R is a 1-dimensional Cohen Macaulay local ring. Consider the minimal reduction $J = (x)$ of $\mathfrak{m} = (x,y,z)$. We see that $y^3 \in J \cap \mathfrak{m}^3 \setminus J\mathfrak{m}^2$, showing that $\mathfrak{m}$ is not 3-standard with respect to J. However, by Remark 6.2, Question 6.1 has a positive answer for $\mathfrak{m}$ and any minimal reduction J of $\mathfrak{m}$.

6.2 3-Standardness of the Maximal Ideal: the Prime Characteristic case

If $(R,\mathfrak{m})$ is a Cohen-Macaulay local ring, we know that $\mathfrak{m}$ is 2-standard by Remark 5.2(2). Example 6.6 shows that the maximal ideal is not 3-standard in general. In this section, we study conditions under which the maximal is 3-standard in the characteristic $p>0$ case. In particular, we prove the Theorem 6.7 and Corollary 6.11. Just as in the Valabrega-Valla Theorem, the graded ring associated to the maximal ideal plays a role in these theorems.

Theorem 6.7. *Let G be a standard graded algebra over a perfect field k of positive characteristic p. Let $x_1,\dots,x_d$ be a linear system of parameters in G. If G is an normal domain, then $H_1(x_1,\dots,x_k;G)_{\leq 2}=0$ for $1\leq k\leq d$.*

Proof. By the colon-capturing property of tight closure, (cf. [7], Theorem 3.1), we see that $(x_1,\dots,x_{k-1}):_G x_k \subseteq (x_1,\dots,x_{k-1})^*$ for $1\leq k\leq d$.

Let $\mathfrak{m}=G_{>0}$. Now by Theorem 0.9, since $(x_1,\dots,x_{k-1})\subseteq \mathfrak{m}$ but not in $\mathfrak{m}^2$, we get $(x_1,\dots,x_{k-1})^*\subseteq (x_1,\dots,x_{k-1})+\mathfrak{m}^2$ for $1\leq k\leq d$.

Thus $(x_1,\dots,x_{k-1}):_G x_k\subseteq (x_1,\dots,x_{k-1})+\mathfrak{m}^2$ for $1\leq k\leq d$. Hence by Proposition 5.9, $H_1(x_1,\dots,x_k;G)_{\leq 2}=0$ for $1\leq k\leq d$. □

Applying Theorem 6.7 to the associated graded ring, we get the following:

Corollary 6.8. *Let $(R,\mathfrak{m},\mathsf{k})$ be a d-dimensional Cohen-Macaulay local ring of positive characteristic p, with a perfect residue field k. Let $G=\mathrm{gr}_R(\mathfrak{m})=R/\mathfrak{m}\oplus\mathfrak{m}/\mathfrak{m}^2\oplus\cdots$ be the associated graded ring of the maximal ideal. Let $J=(x_1,\dots,x_d)$ be a minimal reduction of $\mathfrak{m}$. If G is a normal domain, then $J\cap\mathfrak{m}^3=J\mathfrak{m}^2$.*

Proof. Let s' denote the leading form in G of an element s in R. By Theorem 6.7, $H_1(x_1,\ldots,x_k;G)_{\leq 2}=0$ for $1\leq k\leq d$. Therefore, by Proposition 5.10, $\mathfrak{m}$ is 3-standard with respect to J, i.e., $J\cap\mathfrak{m}^3=J\mathfrak{m}^2$. $\square$

We see in Theorem 6.7 that if G is a normal domain, then $H_1(x_1,\ldots,x_k;G)_{\leq 2}=0$ for $1\leq k\leq d$. However, it is possible that $H_1(x_1,\ldots,x_k;G)_3\neq 0$ even when G is a normal domain as can be seen from the following example.

Example 6.9. Let $R=\mathsf{k}[X,Y,Z]/(X^3+Y^3+Z^3)$, where k is a perfect field such that $\operatorname{char}(\mathsf{k})\neq 3$. Then R is a domain since the polynomial $X^3+Y^3+Z^3$ is irreducible in $\mathsf{k}[X,Y,Z]$. By the Jacobian criterion (e.g., see [21], Theorem 4.4.9), since k is perfect, we see that R is integrally closed in its field of fractions.

Let $G=R[(x,y,z)t]$, the Rees ring associated to the homogeneous maximal ideal $\mathfrak{m}=(x,y,z)$. Since R is a domain and $G\subseteq R[t]$, G is a domain. Moreover, since R is normal and $\mathfrak{m}$ is an integrally closed ideal in R, G is integrally closed in its field of fractions (e.g., see [21], Proposition 5.2.4).

One can see using a computer algebra package that a presentation for G is the following: $G\simeq\mathsf{k}[X,Y,Z,U,V,W]/I$ where $I=(X^3+Y^3+Z^3, X^2U+Y^2V+Z^2W, XU^2+YV^2+ZW^2, U^3+V^3+W^3, YW-ZV, XW-ZU, XV-YU)$. We use lower case letters to denote elements of G.

Consider the linear system of parameters $f_1=x, f_2=y+u$ and $f_3=z+v$. Then $\overline{(x^2-yv+w^2, y^2-zw, z^2)}\in H_1(\underline{f})_3$ is a non-zero element, showing that $H_1(\underline{f})_3\neq 0$.

As a consequence of Proposition 5.10, one observes that if G is the associated graded ring of a Cohen-Macaulay local ring $(R,\mathfrak{m})$, then $J\cap\mathfrak{m}^4\neq J\mathfrak{m}^3$, where $J=(f_1,f_2,f_3)$ is a minimal reduction of $\mathfrak{m}$ such that the leading forms of f_1, f_2 and f_3 in G are x, $y+u$ and $z+v$ respectively.

Theorem 6.10. *Let G be a standard graded k-algebra where k is an algebraically closed field of characteristic $p > 0$. Let $x_1, \ldots, x_d$ be a linear system of parameters in G. If G is reduced and connected in codimension 1, then $H_1(x_1, \ldots, x_k; G)_2 = 0$ for $1 \leq k \leq d$.*

Proof. Let $\operatorname{Min}(G) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$ be the set of minimal primes of G. By Proposition 5.8, it is enough to show that $H_1(x_1, \ldots, x_k; G/\mathfrak{p}_i)_2 = 0$ for $i = 1, \ldots, l$.

Let $\mathfrak{p}$ be a minimal prime of G. Let $(r_1, \ldots, r_k) \in H_1(x_1, \ldots, x_k; G/\mathfrak{p})_2$. Then we have $\sum_{i=1}^{j} r_i x_i = 0$ in $G/\mathfrak{p}$, where $\deg(r_i) = 1$.

Let $\mathfrak{G} = (G/\mathfrak{p})^{\mathrm{gr}+}$ be the graded absolute integral closure of $G/\mathfrak{p}$. Then $\mathfrak{G}$ is a big Cohen-Macaulay $G/\mathfrak{p}$-algebra by Theorem 0.10. Hence $x_1, \ldots, x_d$ form a regular sequence in $\mathfrak{G}$. Therefore the only relations on $x_1, \ldots, x_k$ are the Koszul relations, i.e., we can write $(r_1, \ldots, r_k) = (x_1, \ldots, x_k) S_{k \times k}$, where S is a $k \times k$ skew-symmetric matrix with entries in $\mathfrak{G}$. By degree arguments, we can assume that the entries of S are units in $\mathfrak{G}$, i.e., the entries of S are in k (since k is algebraically closed) and hence in $G/\mathfrak{p}$.

Thus we can write $(r_1, \ldots, r_k)$ in terms of the Koszul relations on $(x_1, \ldots, x_k)$ in $G/\mathfrak{p}$. Thus $\overline{(r_1, \ldots, r_k)} = 0$ in $H_1(x_1, \ldots, x_k; G/\mathfrak{p})$ finishing the proof. □

Corollary 6.11. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a d-dimensional Cohen-Macaulay local ring of positive characteristic p, with an algebraically closed residue field k. Let $G = \mathrm{gr}_R(\mathfrak{m}) = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \cdots$ be the associated graded ring of the maximal ideal. Let $J = (x_1, \ldots, x_d)$ be a minimal reduction of $\mathfrak{m}$. If G is reduced and connected in codimension 1, then $J \cap \mathfrak{m}^3 = J\mathfrak{m}^2$.*

Proof. It is enough to show that $H_1(x'_1, \ldots, x'_k; G)_2 = 0$ for $1 \leq k \leq d$ by Proposition 5.10 and Corollary 5.12. This follows immediately from Theorem 6.10.

□

6.3 The Method of Reduction to Prime Characteristic in Action

Most of the material in this section can be found in [9], sections 2.1 and 2.3. We begin by recalling the following definition (Definition 2.3.1) from [9].

Definition 6.12. *We say that a* k*-algebra* R *is an absolute domain if* $R \otimes_{\mathsf{k}} \bar{\mathsf{k}}$ *is a domain, where* $\bar{\mathsf{k}}$ *is the algebraic closure of* k*. We say that a prime ideal* $\mathfrak{p} \subseteq R$ *is an absolute prime if* $R/\mathfrak{p}$ *is an absolute domain.*

Setup 6.13. *Let* k *be a field of characteristic 0. Let* G *be a standard graded* k*-algebra and* $x_1, \ldots, x_d \in G$ *be a linear system of parameters such that for each* $k = 1, \ldots, d$*,*

$$(x_1, \ldots, x_{k-1}) :_G x_k \subseteq (x_1, \ldots, x_{k-1}) + G_{\geq n}.$$

We will now apply the method of reduction to prime characteristic to this setup. The following lemma ([9], 2.1.4) plays a key role in this process.

Lemma 6.14 (Generic Freeness)**.** *Let* A *be a Noetherian domain,* R *a finitely generated* A*-algebra,* S *a finitely generated* R*-algebra,* W *a finitely generated* S*-module,* M *a finitely generated* R*-submodule of* W *and* N *a finitely generated* A*-submodule of* W*. Let* $V = W/(M+N)$*. Then there exists an element* $a \in A \setminus \{0\}$ *such that* V_a *is free over* A_a*.*

Write $G \simeq \mathsf{k}[X_1, \ldots, X_m]/(F_1, \ldots, F_n)$ where $X_i \mapsto x_i$ for $i = 1, \ldots, d \leq m$. Write $A = \mathbb{Z}[\text{coefficients of the } F_j\text{'s}]$.

Let $G_A = A[X_1, \ldots, X_m]/(F_1, \ldots, F_n)$. By the lemma of Generic Freeness, after inverting an element a of A, and replacing A_a by A, we may assume that G_A is a free A-module.

Since G_A is a free A-module, the inclusion $A \hookrightarrow \mathsf{k}$ induces the injective map $G_A \hookrightarrow G_{\mathsf{k}} := G_A \otimes_A \mathsf{k}$.[1] Further, we see that $G \simeq \mathsf{k}[\underline{X}] \otimes_{A[\underline{X}]} A[\underline{X}]/(\underline{F}) \simeq (\mathsf{k} \otimes_A A[\underline{X}]) \otimes_{A[\underline{X}]} A[\underline{X}]/(\underline{F}) \simeq \mathsf{k} \otimes_A A[\underline{X}]/(\underline{F})$

By further inverting another element of A if necessary (and calling the localization A again), we see by [9], 2.1.14(a)-(c),(g) that for each $k = 1, \ldots, d$, $(x_1, \ldots, x_{k-1}) :_{G_A} x_k \subseteq (x_1, \ldots, x_{k-1}) + (G_A)_{\geq n}$.

Let $\mathfrak{m}_A$ be any maximal ideal in A. Then there is some prime $p \in \mathfrak{m}_A$. Thus if $G' = G_A/\mathfrak{m}_A G_A$, we see that G' is a standard graded k'-algebra, where k' is a field of characteristic $p > 0$. We say that G *descends to* G' or that G' *descends from* G.

Let x_i' denote the image of x_i in G'. Notice that each x_i' is a linear form in G'. Now, by Theorem 2.3.5(c) in [9], we see that $\dim(G) = \dim(G')$, hence $x_1', \ldots, x_d'$ form a linear system of parameters in G'. The condition that $(x_1', \ldots, x_{k-1}') :_{G'} x_k' \subseteq (x_1', \ldots, x_{k-1}') + (G')_{\geq n}$ holds for each $k = 1, \ldots, d$ for all but finitely many maximal ideals $\mathfrak{m}_A \in A$ by Theorem 2.3.5(g) in [9]. Choose an $\mathfrak{m}_A$ such that $(x_1', \ldots, x_{k-1}') :_{G'} x_k' \subseteq (x_1', \ldots, x_{k-1}') + (G')_{\geq n}$ holds for each $k = 1, \ldots, d$.

Suppose further that G is an absolute domain. By Theorem 2.3.6(c) in [9], we see that for all but finitely many maximal ideals $\mathfrak{m}_A$ in A, $G' = G_A/\mathfrak{m}_A$ is an absolute domain. Choosing one such $\mathfrak{m}_A$ for which the condition $(x_1', \ldots, x_{k-1}') :_{G'} x_k' \subseteq (x_1', \ldots, x_{k-1}') + (G')_{\geq n}$ also holds for each $k = 1, \ldots, d$ we see that:

Theorem 6.15. *Let the notation be as in Setup 6.13. Suppose G is an absolute domain. Then there is a field k' of prime characteristic, an absolute domain G' which is a standard graded k'-algebra, $x_1', \ldots, x_d'$, a linear system of parameters in G' satisfying $(x_1', \ldots, x_{k-1}') :_{G'} x_k' \subseteq (x_1', \ldots, x_{k-1}') + (G')_{\geq n}$ for each $k = 1, \ldots, d$ such that G descends to G'.*

[1] We only need that G_A is A-flat.

6.4 The Characteristic Zero Case: Reduction to Prime Characteristic

In this section, we prove an analogue of Theorem 6.10 in the case when the residue field has characteristic zero. We use the method of reduction to characteristic p. Our main source for this technique are sections 2.1 and 2.3 of [9].

Theorem 6.16. *Let G be a standard graded algebra over a field k. Let $x_1,\dots,x_d$ be a linear system of parameters in G. If G is an absolute domain, then $H_1(x_1,\dots,x_k;G)_2 = 0$ for $1 \le k \le d$.*

Proof.

Case(i): Suppose $\mathrm{char}(\mathsf{k}) = p > 0$. Since G is an absolute domain, $G' = G \otimes_{\mathsf{k}} \bar{\mathsf{k}}$ is a domain, where $\bar{\mathsf{k}}$ is the algebraic closure of k. A domain is reduced and connected in codimension 1, hence it follows immediately from Theorem 6.10 that $H_1(x_1,\dots,x_k;G')_2 = 0$ for $1 \le k \le d$. Thus $H_1(x_1,\dots,x_k;G)_2 = 0$ for $1 \le k \le d$.

Case(ii): Suppose $\mathrm{char}(\mathsf{k}) = 0$ and $H_1(x_1,\dots,x_k;G)_2 \neq 0$ for some k, $1 \le k \le d$. By Proposition 5.9 and Theorem 6.15, there is field k' of some positive characteristic $p > 0$, a standard graded k'-algebra G' which is an absolute domain with a system of parameters $x_1,\dots,x_d$ such that $H_1(x_1,\dots,x_k;G')_2 \neq 0$. This contradicts case(i) proving case (ii). $\square$

As a consequence of Theorem 6.16, Proposition 5.10 and Corollary 5.12, we conclude that:

Corollary 6.17. *If $(R,\mathfrak{m},\mathsf{k})$ is a d-dimensional Cohen-Macaulay local ring with associated graded ring $G = \mathrm{gr}_R(\mathfrak{m}) = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \cdots$ and $J = (x_1,\dots,x_d)$ is a minimal reduction of $\mathfrak{m}$, then $J \cap \mathfrak{m}^3 = J\mathfrak{m}^2$ when G is an absolute domain.*

Remark 6.18. Thus we see that with notation as in the above corollary, if G is an absolute domain, then $J \cap \mathfrak{m}^3 = J\mathfrak{m}^2$ for every minimal reduction J of $\mathfrak{m}$. Thus $\mathfrak{m}$ is 3-standard. As a consequence, we see that in this case if R is Cohen-Macaulay with an infinite residue field, the formula $\lambda(\mathfrak{m}^4/J\mathfrak{m}^3) = e_0(\mathfrak{m}) + \sum_{i=0}^{3}(-1)^{i+1}\binom{d-1}{i}\lambda(\mathfrak{m}^{3-i}/\mathfrak{m}^{4-i})$. holds for every minimal reduction J of $\mathfrak{m}$. Thus in this case $\lambda(\mathfrak{m}^4/J\mathfrak{m}^3)$ is independent of the minimal reduction of $\mathfrak{m}$.

References

[1] H. Ananthnarayan, *The Gorenstein Colength of an Artinian Local Ring,* Journal of Algebra, **320** (2008), 3438-3446. Cited on xi, 26

[2] H. Ananthnarayan, *Computing Gorenstein Colength,* Journal of Commutative Algebra (to appear). Cited on x, 36

[3] L. Avramov and G. Levin, *Factoring out the socle of a Gorenstein ring*, J. Algebra, **55** (1978), 7483. Cited on 33

[4] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics **39** (1993), Cambridge University Press. Cited on 3

[5] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, **150** (1994), Springer-Verlag, New York. Cited on 3

[6] R. Fröberg and D. Laksov, *Compressed Algebras,* Conference on Complete Intersections in Acireale, Lecture Notes in Mathematics, **No. 1092** (1984), 121-151, Springer Verlag. Cited on 48

[7] C. Huneke, *Tight Closure and its Applications,* CBMS Lecture Notes **88**, American Mathematical Society, 1996. Cited on xiii, 84

[8] M. Hochster, C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras,* Annals Math., **135** (1992), 53-89. Cited on xiii, xiv

[9] M. Hochster, C. Huneke, *Tight closure in equal characteristic zero*, preprint on http://www.math.lsa.umich.edu/˜hochster/tcz.ps, 2000. Cited on xiv, 87, 88, 89

[10] C. Huneke, B. Ulrich, *Liaison of Monomial Ideals,* Bull. London Math. Soc., **39** (2007), 384-392. Cited on 47

[11] C. Huneke, A. Vraciu, *Special tight closure,* Nagoya Math. J. **170** (2003) 175–183. Cited on xiv

[12] C. Huneke, A. Vraciu, *Rings which are almost Gorenstein,* Pacific J. Math., **225** (2006) no. 1, 85 - 102. Cited on x, xi, 23, 25, 27

[13] A. Iarrobino, *Compressed Algebras: Artin algebras having given socle degrees and maximal length,* Trans. Amer. Math. Soc., **285** (1984), 337-378. Cited on 48

[14] A. Iarrobino, V. Kanev, *Power sums, Gorenstein algebras and determinantal loci*, Lecture Notes in Math., 1721, Springer-Verlag, Berlin, 1999. Cited on 48

[15] B. Johnston, J. K. Verma, *On the length formula of Hoskin and Deligne and associated graded rings of two-dimensional regular local rings,* Math. Proc. Cambridge Phil. Soc., **111** (1992), 423 - 432. Cited on 51, 52

[16] J. Kleppe, J. Migliore, R.M. Miró-Roig, U. Nagel, C. Peterson, *Gorenstein Liaison, Complete Intersection Liaison Invariants and Unobstructedness,* Mem. Amer. Math. Soc., **154** (2001), no. 732, viii+116 pp. Cited on 35, 47

[17] D. G. Northcott, *Injective Modules and Inverse Polynomials,* J. London Math. Soc. (2), **8** (1974), 290-296. Cited on 9, 37

[18] T. Puthenpurakal, *Invariance of a Length Associated to a Reduction,* Communications in Algebra, **33** (2005), 2039-2042. Cited on xii, 81

[19] L. Ried, L. Roberts, M. Roitman, *On complete intersections and their Hilbert functions,* Canadian Mathematical Bulletin, **34** (1991), 525-535. Cited on x, 35, 39

[20] M. Rossi, *Primary ideals with good associated graded ring ,* Journal of Pure and Applied Algebra, **145** (2000), 75-90. Cited on viii, xii, 72

[21] I. Swanson, C. Huneke, *Integral closure of ideals, rings, and modules,* London Math. Soc. Lecture Note Ser., **336** (2006), Cambridge Univ. Press, Cambridge. Cited on xiii, 16, 50, 73, 81, 85

[22] W. Teter, *Rings which are a factor of a Gorenstein ring by its socle,* Invent. Math., **23** (1974), 153-162. Cited on x

[23] P. Valabrega, G. Valla, *Form Rings and Regular Sequences*, Nagoya Math. J., **72** (1978), 93-101. Cited on xii, 72, 78

[24] G. Valla, *Problems and results on Hilbert functions of graded algebras*, Six Lectures in Commutative Algebra, Progress in Mathematics, **166** (1998), BIR, Basel. Cited on 80

List of Symbols